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Clustering of Non-Identical Clocks

Krzysztof Czołczyński, Przemysław Perlikowski,
Andrzej Stefański and Tomasz Kapitaniak

*Division of Dynamics, Technical University of Lodz,
Stefanowskiego 1/15, 90-924 Lodz, Poland*

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We study synchronization of a number of different pendulum clocks hanging from a horizontal beam which can roll on the parallel surface. The results previously obtained for n identical clocks [Czolczynski et al., Prog. Theor. Phys. **122** (2009), 1027] are generalized for the case of non-identical clocks. Pendula have the same period of oscillations so the clocks are accurate but have different masses. It has been shown that after a transient, different types of synchronization between pendula can be observed; (i) the complete synchronization in which all pendula behave identically, (ii) pendula create three or five clusters of synchronized pendula. Contrary to the case of identical clocks antiphase synchronization in pairs is not robust for an even number of clocks. We derive the equations for the estimation of the phase differences between phase synchronized clusters. The evidence, why other configurations with a different number of clusters are not observed, is given.

Subject Index: 034

§1. Introduction

The problem of the synchronization of clocks can be traced back to the Dutch researcher Christian Huygens in the 17th century.[1)–4)] He showed that a couple of mechanical clocks hanging from a common support had been synchronized. Huygens had found the pendulum clocks swung in exactly the same frequency and out of phase, i.e., in antiphase synchronization (phase difference equals π). After the external perturbation, the antiphase state was restored within half an hour and remained indefinitely.

Recently, Huygens' experiment has attracted increasing attention from different research groups.[5)–14)] Pogromsky et al.[5)] designed a controller for synchronization problem for two pendula suspended on an elastically supported rigid beam. To explain Huygens' observations Bennett et al.[6)] built an experimental device consisting of two interacting pendulum clocks hanged on a heavy support which was mounted on a low-friction wheeled cart. The device moves by the action of the reaction forces generated by the swing of two pendula and the interaction of the clocks occurs due to the motion of the clocks' base. It has been shown that to repeat Huygens' results, the high precision (the precision that Huygens certainly could not achieve) is necessary. Senator[7)] developed a qualitative approximate theory of clocks' synchronization. This theory explicitly includes the essential nonlinear elements of Huygens' system, i.e., escapement mechanisms but also includes many simplifications. An interaction mechanism between two oscillators leading to exact antiphase and in-phase synchronization has been described by Dilao.[8)] It has been shown that if two coupled nonlinear oscillators reach the antiphase or the in-phase synchronization, the oscillation frequency is different from the frequency of the uncoupled oscillators.

A device mimicking Huygens' clock experiment, the so-called "coupled pendula of the Kumamoto University",[9] consists of two pendula whose suspension rods are connected by a weak spring, and one of the pendula is excited by an external rotor. The numerical results of Fradkov and Andrievsky[10] show simultaneous approximate in-phase and antiphase synchronization. Both types of synchronization can be obtained for different initial conditions. Additionally, it has been shown that for small difference in the pendula frequencies they may not synchronize.

A very simple demonstration device was built by Pantaleone.[11] It consists of two metronomes located on a freely moving light wooden base. The base lies on two empty soda cans which smoothly roll on the table. Both in-phase and antiphase synchronizations of the metronomes have been observed. Recently, Ulrichs et al.[12] have studied synchronization scenarios of coupled mechanical metronomes showing the onset of synchronization for two, three, and 100 globally coupled metronomes.

In the previous papers[12],[13] we studied a synchronization problem for n identical pendulum clocks hanging from an elastically fixed horizontal beam. It was assumed that each pendulum performs a periodic motion which starts from different initial conditions. We showed that after a transient different types of synchronization between pendula can be observed. The first type is in-phase complete synchronization in which all pendula behave identically. In the second type one can identify the groups (clusters) of synchronized pendula. We showed that only configurations of three and five clusters are possible and derive algebraic equations for the phase difference between the pendula in different clusters. In the third type, which is possible only for even n, one observes anti-phase synchronization in $n/2$ pairs of pendula. Besides these synchronized states it is possible to observe the uncorrelated motion of the pendula.

In this paper we generalize these results for the case of n non-identical pendulum clocks. It has been assumed that the clocks under consideration are accurate, i.e., show exactly the same time, but can differ by the design of the escapement mechanism and the pendulum. Particularly, we consider the pendula with the same period of oscillations and different masses. Our main result shows that the phase synchronization of non-identical clocks is possible only when three or five clusters are created. Contrary to the case of identical clocks this result holds for both even and odd number of clocks. We derive the equations which allow the estimation of the phase differences between clusters. We argue why other cluster configurations are not possible.

The paper is organized as follows. In §2 we present our theoretical model which describes the dynamics of n coupled non-identical pendulum clocks. Section 3 presents the results of our numerical studies. We present typical examples of stable phase synchronization in the considered system, associated with pendula configurations and derive equations for estimation of the phase shifts between the pendula. Here we give evidence why one can observe configurations of only three or five clusters. Finally, we discuss why the other configurations are not possible and summarize our results in §4.

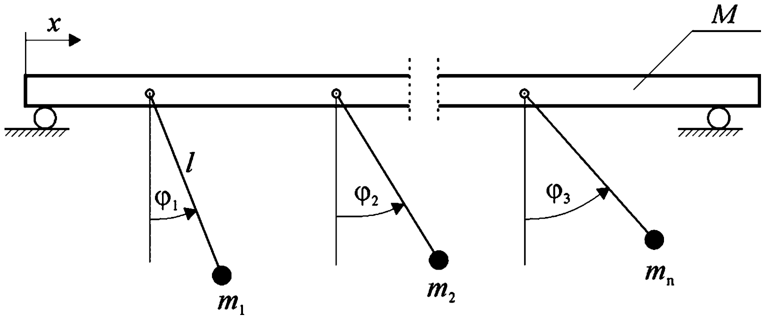

Fig. 1. The model of n pendulum clocks hanging from a horizontal beam.

§2. The model

In the current studies we consider a system shown in Fig. 1. The beam of mass M can move in the horizontal direction x. The beam supports n pendulum clocks with pendula of the same length l, identical period of oscillations T (due to the small amplitudes of the clocks' pendula[15]) and different masses $m_i (i = 1, 2, \ldots)$. Under these assumptions the clocks are accurate, i.e., when uncoupled show exactly the same time but can be different by the design of the pendulum and the escapement mechanism. The position of the i-th pendulum is given by a variable φ_i and its oscillations are damped by the viscous friction described by damping coefficient $c_{\varphi i}$. We assume that this friction is proportional to the pendulum mass m_i, i.e., $c_{\varphi i} = c_\varphi m_i$ and the length of the pendula l is equal to $g/4\pi^2$= 0.2485 [m], where g is a gravitational acceleration. The beam is considered as a rigid body so the elastic waves along it are not considered. We describe the phenomena which take place far below the resonances for both longitudinal and transverse oscillations of the beam.

The system equations can be written in a form of Euler-Lagrange equations:

$$m_i l^2 \ddot{\varphi}_i + m_i \ddot{x} l \cos\varphi_i + c_\varphi m_i \dot{\varphi}_i + m_i g l \sin\varphi_i = M_{Di}, \tag{1}$$

$$\left(M + \sum_{i=1}^{n} m_i\right)\ddot{x} + \sum_{i=1}^{n}\left(m_i l \ddot{\varphi}_i \cos\varphi_i - m_i l \dot{\varphi}_i^2 \sin\varphi_i\right) = 0. \tag{2}$$

The clock escapement mechanism (described in details in 13)) represented by momentum M_{Di} provides the energy needed to compensate the energy dissipation due to the viscous friction $c_{\varphi i}$ and to keep the clocks running.[15] This mechanism acts in two successive steps (the first step is followed by the second one and the second one by the first one). In the first step if $\varphi_i < \gamma_N$ then $M_{Di} = M_{Ni}$ and when $\varphi_i < 0$ then $M_{Di} = 0$, where γ_N and M_{Ni} are constant values which characterize the mechanism. For the second stage one has for $-\gamma_N < \varphi_i < 0$ $M_{Di} = -M_{Ni}$ and for $\varphi_i > 0$ M_{Di} $= 0$. In the undamped ($c_{\varphi i} = 0$) and unforced ($M_{Di} = 0$) case when the beam M is at rest ($x = 0$) each pendulum oscillates with the period T equal to 1.0 [s] and frequency $\alpha = 2\pi$ [s^{-1}]. Under these assumptions the dynamics of the pendulum clock is described by a self-excited oscillator with a limit cycle[16] (see also Ref. 17)). The dynamics of the other type of clock escapement mechanism, i.e., verge and foliot mechanism is described in Refs. 5), 18) and 19).

After the initial transient the pendula perform the periodic oscillations so the solution of Eq. (1) can be approximately described as

$$\varphi_i = \Phi \sin\left(\alpha t + \beta_i\right). \tag{3}$$

Assuming that Φ is small (typically for pendulum clocks $\Phi < 2\pi/36$ and for clocks with long pendula Φ is even smaller[15)]) one can linearize Eq. (2) as follows:

$$\left(M + \sum_{i=1}^{n} m_i\right)\ddot{x} + \sum_{i=1}^{n}\left(m_i l\ddot{\varphi}_i - m_i l\dot{\varphi}_i^2\varphi_i\right) = 0, \tag{4}$$

or substituting Eq. (3) into Eq. (4)

$$\left(M + \sum_{i=1}^{n} m_i\right)\ddot{x} = \sum_{i=1}^{n}\left(m_i l\alpha^2\Phi\sin(\alpha t + \beta_i) + m_i l\alpha^2\Phi^3\cos^2(\alpha t + \beta_i)\sin(\alpha t + \beta_i)\right). \tag{5}$$

Taking into consideration the relation $\cos^2\alpha\sin\alpha = 0.25(\sin\alpha + 3\sin 3\alpha)$ and substituting

$$U = M + \sum_{i=1}^{n} m_i,\ F_{1i} = m_i l\alpha^2(\Phi + 0.25\Phi^3),\ F_{3i} = 0.75 m_i l\alpha^2\Phi^3,$$

one gets

$$U\ddot{x} = \sum_{i=1}^{n}\left(F_{1i}\sin(\alpha t + \beta_i) + F_{3i}\sin(3\alpha t + 3\beta_i)\right). \tag{6}$$

The right-hand side of Eq. (6) represents the force with which n pendula act on the beam M. Equation (6) allows the determination of the beam acceleration $\ddot{x}$ and (after integration) of its velocity $\dot{x}$ and displacement x. Notice that this force consists only of the first and the third harmonics. Later this property will be essential in explanation why in the system (1) and (2) one observes only configurations consisting of three and five clusters of synchronized pendula.

To study the stability of the solution of Eqs. (1) and (2) we add perturbations δ_i and σ to the variables φ_i and x and obtain the following linearized variational equation:

$$m_i l^2\ddot{\delta}_i + m_i\ddot{\sigma}l\cos\varphi_i + m_i l\delta_i(g\cos\varphi_i - \ddot{x}\sin\varphi_i) + c_\phi\dot{\delta}_i = 0, \tag{7}$$

$$M + \sum_{i=1}^{n} m_i\ddot{\sigma} + \sum_{i=1}^{n}(m_i l\delta_i\cos\varphi_i - m_i l\dot{\varphi}_i^2\delta_i\cos\varphi_i - m_i l\dot{\varphi}_i^2\sin\varphi_i - 2m_i l\dot{\varphi}_i\dot{\delta}_i\sin\varphi_i) = 0. \tag{8}$$

The solution of Eqs. (1) and (2) given by $\varphi_i(t)$ and $x(t)$ is stable when the solution of Eqs. (7) and (8) δ_i and σ tend to zero for $t \to \infty$. All the pendula configurations described in this paper fulfil this relation.

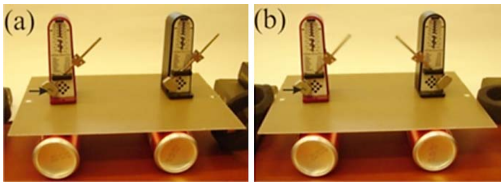

Fig. 2. Two metronomes located on the plate which can roll on the base: (a) complete synchronization, (b) antiphase synchronization. Arrows indicate additional masses added to differentiate the total masses of the metronomes' pendula.

§3. Pendula's configurations

3.1. *Two clocks* ($n = 2$)

In the case of two clocks when their pendula have different masses m_1 and m_2 and the same period of oscillations T one can observe two types of synchronization.[20)] The first one is the complete synchronization when both pendula oscillate in the same way (i.e., $\varphi_1 = \varphi_2$) and move in the opposite direction to the beam motion. In the case of complete synchronization the motion of the beam identically influences the pendula's period of oscillation. The second one is the antiphase synchronization when there is π phase shift between displacements of pendula.

In Figs. 2(a) and (b) we show the simple experimental confirmation of the stability of both synchronization configurations. Two metronomes located on the elastic plate which can roll on the base obtain a complete synchronization (Fig. 2(a)) and an antiphase synchronization (Fig. 2(b)). The masses of metronomes pendula are slightly different as to one of them the small masses (indicated by arrows) have been added. Notice that in the case of antiphase synchronization the plate is not at rest (as in the case of identical pendula), but oscillates with a small amplitude. There is also a small difference in pendula amplitudes.

3.2. *Three clocks* ($n = 3$)

Generally, in the system with three pendulum clocks one can observe the following synchronization cases; (i) complete synchronization, (ii) phase synchronization with the constant phase shifts between pendula, i.e., $\varphi_1 - \varphi_2 = \text{constant}$, $\varphi_2 - \varphi_3 = \text{constant}$, $\varphi_1 - \varphi_3 = \text{constant}$. Antiphase synchronization can be observed only as a special case of (ii) and occurs when the sum of the masses of two pendula (1 and 2) is equal to the mass of the third pendulum (3), say $m_1 + m_2 = m_3$. The first and the second pendula create a cluster ($\varphi_1 = \varphi_2$). Pendula 1 and 2 oscillate in antiphase to the pendulum 3, i.e., $\varphi_1 = \varphi_2 = -\varphi_3$.

In our numerical simulations Eqs. (1) and (2) have been integrated by the Runge-Kutta method. The initial conditions have been set as follows; (i) for the beam $x(0) = \dot{x}(0) = 0$, (ii) for the pendula the initial conditions $\varphi_1(0), \dot{\varphi}_1(0)$ have been calculated from the assumed initial phase differences β_{II} and β_{III} (in all calculations $\beta_{\text{I}} = 0$ has been taken) using Eq. (3), i.e., $\varphi_1(0) = 0$, $\dot{\varphi}_1(0) = \alpha\Phi$, $\varphi_2(0) = \Phi \sin \beta_{\text{II}}$, $\dot{\varphi}_2(0) = \alpha\Phi \cos \beta_{\text{II}}$ (as it will be explained later the angles $\beta_{\text{I}} = \beta_1 = 0$, $\beta_{\text{II}} = \beta_2$, $\beta_{\text{III}} = -\beta_3$ have been introduced for better description of the symmetrical configurations). To prove the stability of the obtained configurations we used Eqs. (7) and (8).

Stable configurations of the pendula can be visualized in the following maps. We plot the position of each pendulum given by Eqs. (1) and (2) (after decay of the transients) in the phase space φ_i, $\dot{\varphi}_i$ at the time when the first pendulum is moving through the equilibrium position $\varphi_1 = 0$ with the positive velocity $\dot{\varphi}_1 > 0$. After the initial transients the pendula perform periodic oscillations, which are visible in such maps by a single point for each pendulum (for better visibility indicated as a black dot). As the pendula oscillate with the same amplitude the distance of each point to the origin $(0, 0)$ is equal. The lines between these points and the origin are equal to the phase differences β_i between the oscillations of the pendula. White circle around the group of pendula indicates that the cluster of synchronized pendula has been created.

The examples of the synchronized states of the system with three non-identical clocks are shown in Figs. 3(a)–(c). Figure 3(a) presents the pendula configuration obtained for beam mass $M= 10.0$ and different pendula masses: $m_1 = 1.0, m_2 = m_3 = 2.0$. As $m_2 = m_3$ and $\beta_{\mathrm{II}} = \beta_{\mathrm{III}}$, one can observe symmetrical phase synchronization. Phase differences $\beta_{\mathrm{II}} = \beta_{\mathrm{III}} = 104.5^\circ$ are different from that obtained for the case of identical pendula masses $m_1 = m_2 = m_3 = 1.0$[11] where we observed $\beta_{\mathrm{II}} = \beta_{\mathrm{III}} = 120^\circ$. In Fig. 3(b) we show the results for: $m_1 = 1.0$, $m_2 = 1.75$, $m_3 = 2.25$. Nonsymmetrical phase synchronization with phase differences: $\beta_{\mathrm{II}} = 73^\circ$, $\beta_{\mathrm{III}} = 132^\circ$ has been observed. Finally, Fig. 3(c) shows the example of the complete synchronization observed for: $m_1 = 1.0$, $m_2 = 1.75$, $m_3 = 2.25$. Different configurations of the system (1) and (2) have been obtained by setting various initial conditions.

Phase differences β_{II} and β_{III} can be approximately estimated on the basis of the linear approximation derived in §2. In the case of three clocks the sum of the forces acting on the beam M (the right-hand side of Eq. (6)) is equal to zero when

$$\begin{aligned} &F_{11}\sin\alpha t + F_{12}\sin\alpha t\cos\beta_{\mathrm{II}} + F_{12}\cos\alpha t\sin\beta_{\mathrm{II}} + F_{13}\sin\alpha t\cos\beta_{\mathrm{III}} \\ &-F_{13}\cos\alpha t\sin\beta_{\mathrm{III}} + F_{31}\sin 3\alpha t + F_{32}\sin 3\alpha t\cos 3\beta_{\mathrm{II}} \\ &+F_{32}\cos 3\alpha t\sin 3\beta_{\mathrm{II}} + F_{33}\sin 3\alpha t\cos 3\beta_{\mathrm{III}} - F_{33}\cos 3\alpha t\sin 3\beta_{\mathrm{III}} = 0. \end{aligned} \tag{9}$$

In Eq. (9) the phase shift of the pendulum 1 has been taken as zero (the reference point on the time axis t). Additionally, due to the relation $\beta_{\mathrm{II}} = \beta_2$ and $\beta_{\mathrm{III}} = -\beta_3$, symmetrical configuration of Fig. 3(a) is better visible as $\beta_{\mathrm{II}} = \beta_{\mathrm{III}}$. After some algebraic manipulations one gets

$$\begin{aligned} &\sin\alpha t(F_{11} + F_{12}\cos\beta_{\mathrm{II}} + F_{13}\cos\beta_{\mathrm{III}}) + \cos\alpha t(F_{12}\sin\beta_{\mathrm{II}} - F_{13}\sin\beta_{\mathrm{III}}) \\ &+\sin 3\alpha t(F_{31} + F_{32}\cos 3\beta_{\mathrm{II}} + F_{33}\cos 3\beta_{\mathrm{III}}) + \cos 3\alpha t(F_{32}\sin 3\beta_{\mathrm{II}} - F_{33}\sin 3\beta_{\mathrm{III}}) = 0. \end{aligned} \tag{10}$$

Equation (10) showing the force acting on the beam M can be expressed as the sum of the first and third harmonics. Equation (10) is fulfilled for phase shifts β_{II} and β_{III}, given by

$$F_{11} + F_{12}\cos\beta_{\mathrm{II}} + F_{13}\cos\beta_{\mathrm{III}} = 0,$$

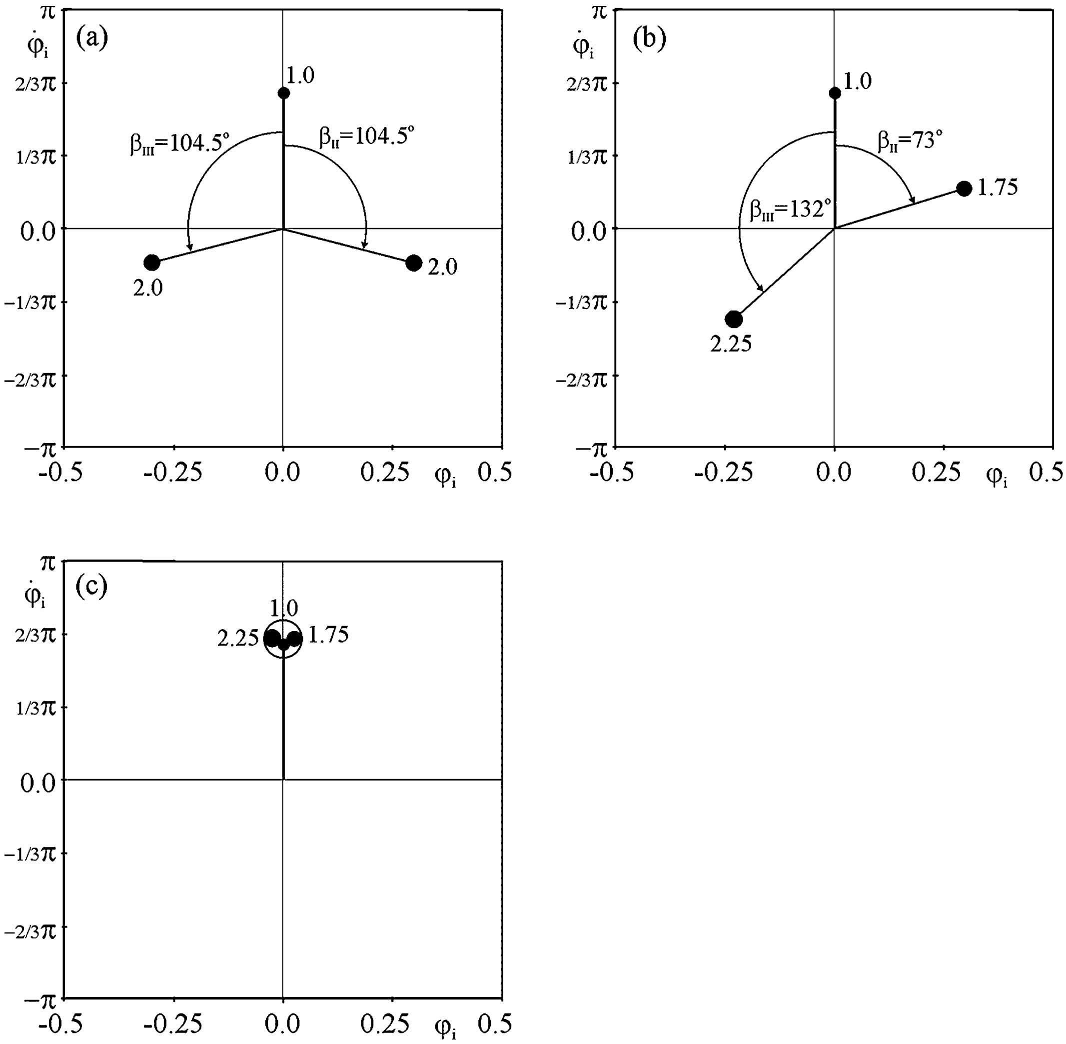

Fig. 3. Synchronization configurations for three pendula; $M = 10.0$; (a) symmetrical synchronization of three pendula $m_1 = 1.0$, $m_2 = m_3 = 2.0$, $\beta_{I0} = 0°$, $\beta_{II0} = 60°$, $\beta_{III0} = 240°$, (b) unsymmetrical synchronization of three pendula $m_1 = 1.0$, $m_2 = 1.75$, $m_3 = 2.25$, $\beta_{I0} = 0°$, $\beta_{II0} = 60°$, $\beta_{III0} = 240°$, (c) full synchronization of three pendula $m_1 = 1.0$, $m_2 = 1.75$, $m_3 = 2.25$, $\beta_{I0} = 0°$, $\beta_{II0} = 25°$, $\beta_{III0} = 305°$.

$$\begin{aligned} &F_{12}\sin\beta_{II} - F_{13}\sin\beta_{III} = 0,\\ &F_{31} + F_{32}\cos 3\beta_{II} + F_{33}\cos 3\beta_{III} = 0,\\ &F_{32}\sin 3\beta_{II} - F_{33}\sin 3\beta_{III} = 0. \end{aligned} \tag{11}$$

Equations (11) have no solution. i.e., for the system with three clocks it is impossible to have both first and third harmonics of the force acting on the beam equal to zero. The stable pendula configuration occurs when the first harmonic is equal to zero, so the phase differences β_{II} and β_{III} can be calculated from the first two equations:

$$F_{11} + F_{12}\cos\beta_{II} + F_{13}\cos\beta_{III} = 0,$$

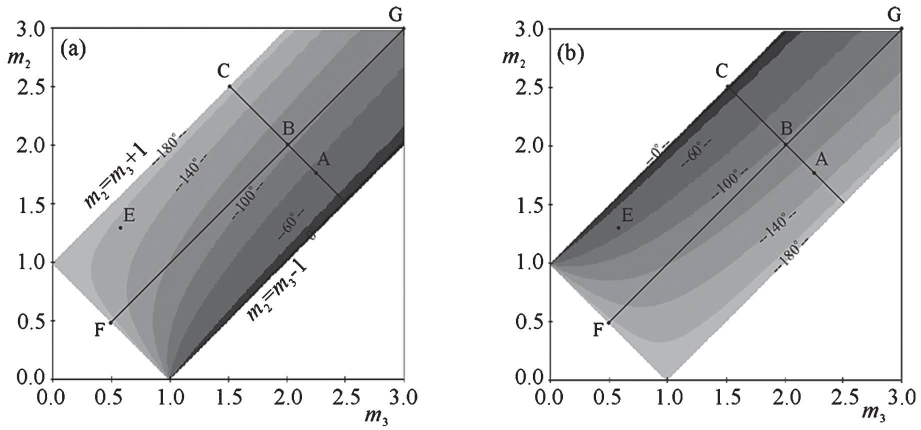

Fig. 4. Phase differences β_{II} and β_{III} versus pendula masses; (a) contour map of β_{II} versus m_2 and m_3, (b) contour map of β_{III} versus m_2 and m_3.

$$F_{12}\sin\beta_{\text{II}} - F_{13}\sin\beta_{\text{III}} = 0. \tag{12}$$

Dividing both sides of Eq. (12) by F_{11} one gets

$$\begin{aligned} m_1 + m_2\cos\beta_{\text{II}} + m_3\cos\beta_{\text{III}} &= 0, \\ m_2\sin\beta_{\text{II}} - m_3\sin\beta_{\text{III}} &= 0. \end{aligned} \tag{13}$$

To solve Eq. (13) we have been searching for the zero minimum of the following function:

$$H_{13} = (m_1 + m_2\cos\beta_{\text{II}} + m_3\cos\beta_{\text{III}})^2 + (m_2\sin\beta_{\text{II}} - m_3\sin\beta_{\text{III}})^2\,. \tag{14}$$

The function H_{13} represents the square of the amplitude of the first harmonic component of the force acting on the beam M. Notice that the phase differences β_{II} and β_{III} calculated from Eq. (14) do not depend on the models of escapement mechanism and friction.

Our calculations show that for $m_1 = 1.0$, $m_2 = 2.0$, $m_3 = 2.0$ the function $H_{13}(\beta_{\text{III}}, \beta_{\text{II}})$ is equal zero for $\beta_{\text{III}} = \beta_{\text{II}} = 104.5°$, i.e., the same values as numerically obtained from the numerical integration of Eqs. (1) and (2) (see Fig. 3(a)). For the parameters of Fig. 3(b) ($m_1 = 1.0$, $m_2 = 1.75$, $m_3 = 2.25$) one gets the same agreement as $H_{13\min}(\beta_{\text{III}}, \beta_{\text{II}}) = H_{13}(132°, 73°) = 0$. Further calculations confirm that the phase synchronization in the system (1) and (2) occurs for the phase difference β_{II} and β_{III} given by Eq. (13) and give evidence that the phase synchronization occurs when the first harmonic component of the force acting on the beam M is equal to zero. In this case the beam M is oscillating with the period three times smaller than the periods of the pendula's oscillations. The motion of the beam influences the oscillations' periods of each pendula in the same way and in the steady state these periods are equal, i.e., the condition for synchronization is fulfilled.

The properties of the solution of Eq. (13) are discussed in Figs. 4(a) and (b). The contour map showing the values of phase differences β_{II} and β_{III} for $m_1 = 1.0$ and different masses of the pendula m_2 and m_3 are shown in Figs. 4(a) and (b).

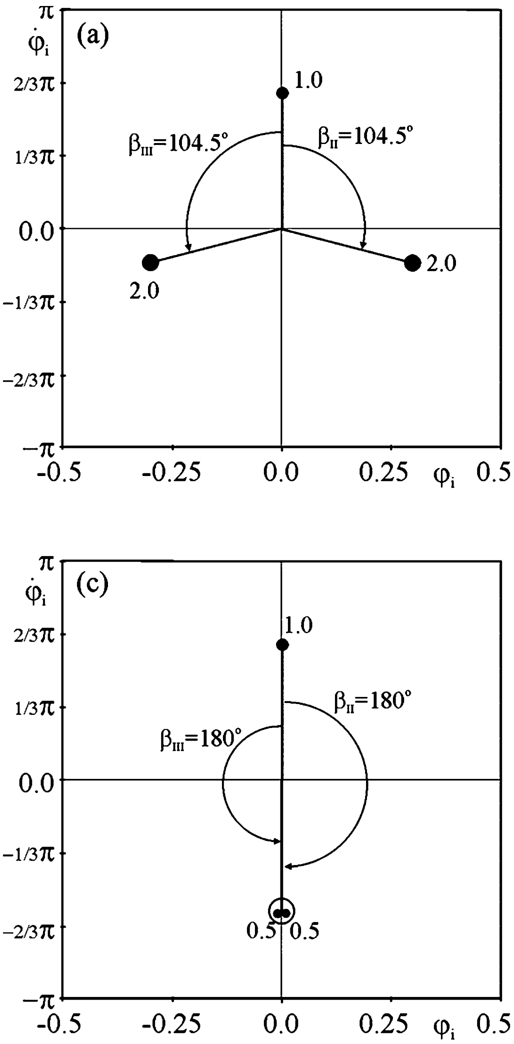

Fig. 5. Synchronization configurations of three pendula for various values of m_2 and m_3; (a) $m_3 = m_2 = 2.0$, (b) $m_3 = 1.5$, $m_2 = 2.5$, (c) $m_3 = m_2 = 0.5$, (d) $m_3 = m_2 = 100.0$.

In Figs. 4(a) and (b) point A ($m_3 = 2.25$, $m_2 = 1.75$) represents the configuration of Fig. 3(b) ($\beta_{\rm II} = 73°$ and $\beta_{\rm III} = 132°$) and point B ($m_3 = 2.0$, $m_2 = 2.0$ and $\beta_{\rm II} = \beta_{\rm III} = 104.5°$) – this configuration shown in Fig. 5(a). Notice that starting from the symmetrical configuration (point B) due to the simultaneous decrease of the value of m_3 and increase of the value of m_2, phase difference $\beta_{\rm II}$ tends to 180° and phase difference $\beta_{\rm III}$ tends to zero. In the limit case – point C – pendula mass $m_1 = 1.0$ and $m_3 = 1.5$ create a cluster oscillating in antiphase to the pendulum mass $m_2 = 2.5$ (equal to the mass of the created cluster) as can be seen in Fig. 5(b). When the mass m_2 is larger than the sum of the masses $m_1 + m_3$ (white region in Figs. 4(a) and (b)) Eq. (11) has no solution and phase synchronization is not observed. Let us start again from the symmetrical configuration of Fig. 3(a) ($m_2 = m_3 = 2.0$) and decrease the values of $m_2 = m_3$ and observe the values of phase differences

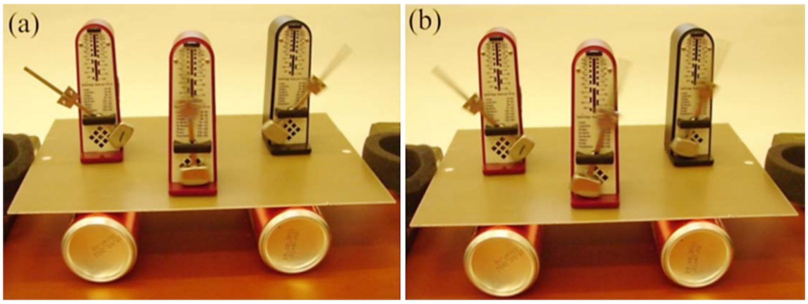

Fig. 6. Three metronomes located on the plate which can roll on the base: (a) symmetrical synchronization with phase shift $\beta_{II} = \beta_{III} = 120°$, (b) antiphase synchronization of the left metronome with the cluster of the center and right metronomes (the mass of the left pendulum is equal to the sum of the masses of the center and right pendula).

$\beta_{II} = \beta_{III}$ increase towards 180°. In the limit case (point F) pendula mass $m_2 = 0.5$ and $m_3 = 0.5$ create a cluster which oscillates in antiphase with pendulum mass $m_1 = 1.0$ as shown in Fig. 5(c). On the other hand, with the increase of the values of $m_2 = m_3$, phase differences $\beta_{II} = \beta_{III}$ decrease (for example to 98.5° for $m_2 = m_3 = 3.0$ – point G in Figs. 4(a) and (b)). In the limit as $m_2 = m_3$ tends to infinity, $\beta_{II} = \beta_{III}$ tends to 90° – the corresponding configuration is shown in Fig. 5(d).

One can conclude that the phase synchronization can occur when the mass of the largest pendulum is smaller than the sum of the masses of the other two pendula. If $m_1 > m_2$ and $m_1 > m_3$ one gets

$$m_1 \leq m_2 + m_3. \tag{15}$$

Relation (15) gives the necessary condition for the phase synchronization in the system with three pendulum clocks.

Typical configurations for the system of three clocks have been also observed in the experiments with metronomes described in Figs. 6(a) and (b). Three metronomes (with different masses of pendula) located on the plate which can roll on the base can show the symmetrical synchronization with phase shift $\beta_{II} = \beta_{III} = 120°$ (Fig. 6(a)) and antiphase synchronization of the left metronome with the cluster consisting of the center and right metronomes (Fig. 6(b)). In the second case the mass of the left pendulum is equal to the sum of the masses of the center and right pendula.

Notice that the method of the phase shift estimation (Eqs. (9)–(14)) and particularly necessary condition (15), derived for three pendula, can be generalized to any number of pendula synchronized in three clusters. In this case one can rewrite condition (15) in the form:

$$\bar{m}_1 \leq \bar{m}_2 + \bar{m}_3, \tag{16}$$

where $\bar{m}_1$, $\bar{m}_2$ and $\bar{m}_3$ are respectively the sum of pendula's masses in first, second and third cluster.

3.3. *Four clocks* ($n = 4$)

In the system with four pendulum clocks one can observe two types of synchronization; (i) the complete synchronization of all pendula (pendula oscillate in

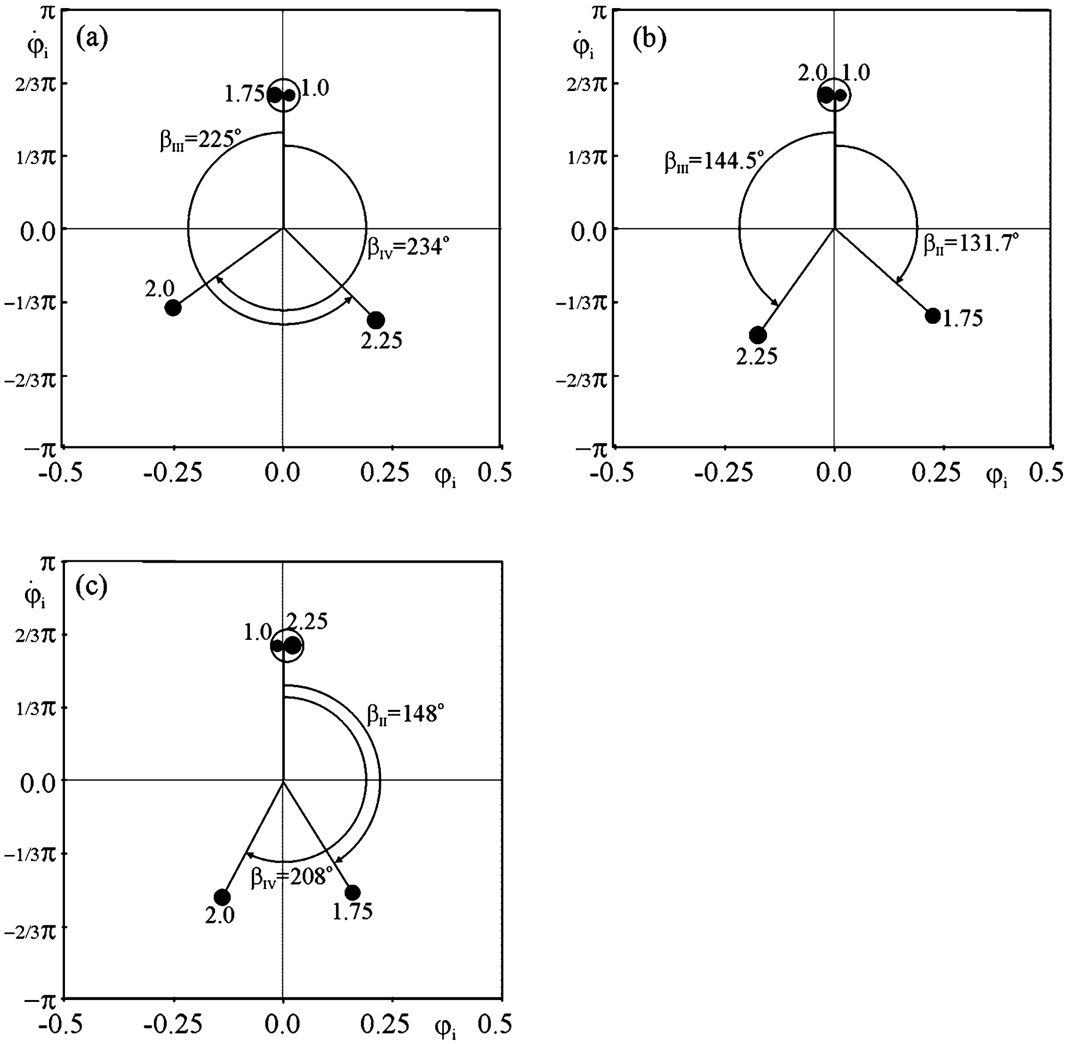

Fig. 7. Synchronization configurations for four pendula: $M = 10.0$; (a) pendula $m_1 = 1.0$, $m_2 = 1.75$ form the cluster, $\beta_{I0} = 0°$, $\beta_{II0} = 73°$, $\beta_{III0} = 130°$, $\beta_{IV0} = 270°$, (b) pendula $m_1 = 1.0$, $m_4 = 2.0$ form the cluster, $\beta_{I0} = 0°$, $\beta_{II0} = 73°$, $\beta_{III0} = 130°$, $\beta_{IV0} = 340°$, (c) pendula $m_1 = 1.0$, $m_3 = 2.25$ form the cluster, $\beta_{I0} = 0°$, $\beta_{II0} = 140°$, $\beta_{III0} = 340°$, $\beta_{IV0} = 220°$.

antiphase with the oscillations of the beam), (ii) the phase synchronization between a cluster of two synchronized pendula and two other pendula (two clusters with one pendulum). In the case of phase synchronization different pendula can create the cluster (there are six different possibilities: $1 + 2$, $1 + 3$, $1 + 4$, $2 + 3$, $2 + 4$, $3 + 4$). The necessary condition for the existence of particular configuration (16) states: the mass of the largest cluster (with one or two pendula) has to be smaller than the sum of other two clusters masses.

Consider a few examples of the phase synchronization for the beam mass $M = 10.0$ and four pendula mass $m_1 = 1.0$, $m_2 = 1.75$, $m_3 = 2.25$, $m_4 = 2.0$. For these parameters the above condition of the existence of the cluster (16) is fulfilled by three pairs of pendula: $1+2$, $1+3$ and $1+4$. The corresponding configurations are shown in Figs. 7(a)–(c). The same phase differences as observed in Figs. 7(a)–(c) can be calculated from the function H_{13} given by Eq. (14). Notice that in this case cluster has to be considered as a single pendulum with mass equal to the total mass of the pendula in cluster.

3.4. *Five clocks* ($n = 5$)

In the system with five pendulum clocks we observed three different types of synchronization: (i) complete synchronization of all pendula, (ii) phase synchronization of three clusters and (iii) phase synchronization of five pendula.

The examples of synchronization configurations for the system with five pendulum clocks are shown in Figs. 8(a)–(d). Figure 8(a) presents the results obtained for: $M = 10.0$, $m_1 = m_2 = m_3 = 1.0$, $m_4 = 0.75$, $m_5 = 1.25$. Observe the phase synchronization with the following phase differences: $\beta_{\mathrm{II}} = 124.5°$, $\beta_{\mathrm{III}} = 162°$, $\beta_{\mathrm{IV}} = 71°$, $\beta_{\mathrm{V}} = 77°$. In this configuration the beam M is in rest. Notice that contrary to the case of identical clocks[11),12)] the obtained configuration is unsymmetrical. The configuration obtained for the same parameter values but different initial phases is shown in Fig. 8(b). The two clusters with masses: $(m_1 + m_4) = 1.75$, $(m_2 + m_3) = 2.0$ and pendulum 5 ($m_5 = 1.25$) are phase synchronized. The phase differences are respectively $\beta_{\mathrm{II}} = 142°$ and $\beta_{\mathrm{III}} = 98°$. Different three clusters' configurations are shown in Figs. 6(c) and (d). Figure 8(c) shows the configuration of the clusters which consist respectively of pendula 1 and 3 ($m_1 + m_3 = 3$), pendulum 4 ($m_4 = 0.75$) and pendula 2 and 5 ($m_2 + m_5 = 2.25$). The phase differences between clusters are given by $\beta_{\mathrm{II}} = 80°$ and $\beta_{\mathrm{III}} = 161°$. The configuration of the clusters consisting of pendulum 1 ($m_1 = 1.0$), pendula 2 and 4 ($m_2 + m_4 = 1.75$) and pendula 3 and 5 ($m_3 + m_5 = 2.25$) with phase differences $\beta_{\mathrm{II}} = 72°$ and $\beta_{\mathrm{III}} = 133°$ is shown in Fig. 8(d). The last possible configuration with clusters consisting of pendulum 1 ($m_1 = 1.0$), pendula 2 and 4 ($m_2 + m_4 = 2.0$), pendula 3 and 5 ($m_3 + m_5 = 2.0$) and phase differences $\beta_{\mathrm{II}} = 104.5°$ and $\beta_{\mathrm{III}} = 104.5°$ has been already shown in Fig. 3(a). Other three cluster configurations are either equivalent to these presented in Figs. 8(a)–(c) and 3(a) or do not fulfill condition (16). (In this case $\bar{m}_1$, $\bar{m}_2$ and $\bar{m}_3$ indicate the total masses of each of three clusters.)

Phase differences β_{II}, $\beta_{\mathrm{III}}, \beta_{\mathrm{IV}}$ and β_{V} can be approximately estimated on the basis of the linear approximation derived in §2. In the case of five pendulum clocks the forces acting on the beam M are equal to zero when

$$
\begin{aligned}
&F_{11} + F_{12}\cos\beta_{\mathrm{II}} + F_{13}\cos\beta_{\mathrm{III}} + F_{14}\cos\beta_{\mathrm{IV}} + F_{15}\cos\beta_{\mathrm{V}} = 0,\\
&F_{12}\sin\beta_{\mathrm{II}} - F_{13}\sin\beta_{\mathrm{III}} + F_{14}\sin\beta_{\mathrm{IV}} - F_{15}\sin\beta_{\mathrm{V}} = 0,\\
&F_{31} + F_{32}\cos 3\beta_{\mathrm{II}} + F_{33}\cos 3\beta_{\mathrm{III}} + F_{34}\cos 3\beta_{\mathrm{IV}} + F_{35}\cos 3\beta_{\mathrm{V}} = 0,\\
&F_{32}\sin 3\beta_{\mathrm{II}} - F_{33}\sin 3\beta_{\mathrm{III}} + F_{34}\sin 3\beta_{\mathrm{IV}} - F_{35}\sin 3\beta_{\mathrm{V}} = 0.
\end{aligned} \tag{17}
$$

Contrary to the case with three clocks (11) now we have four equations with four unknown phase differences β_{II}, β_{III}, β_{IV} and β_{V} and it is possible to find the solution which fulfills all equations of (17). For such values of β_{II}, β_{III}, β_{IV} and β_{V} both the first and the third harmonic of the force acting on the beam M vanish and the beam

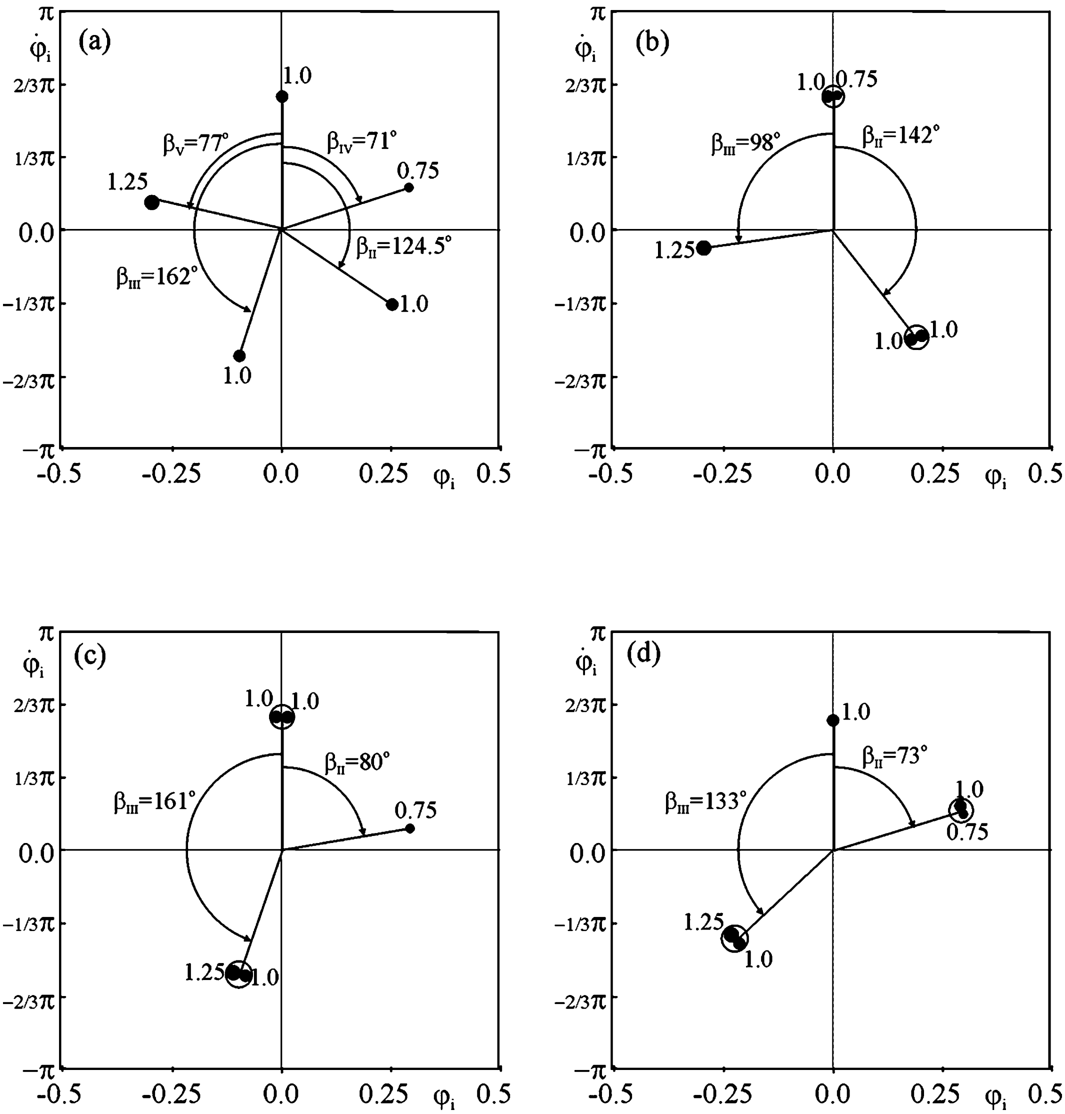

Fig. 8. Synchronization configurations of five pendula: $M = 10.0$; $m_1 = m_2 = m_3 = 1.0$, $m_4 = 0.75$, $m_5 = 1.25$; (a) configuration of five clusters, (b) configuration of three clusters mass 1.75, 2.0, 1.25, (c) configuration of three clusters of the following masses 2.0, 0.75, 2.25, (d) configuration of three clusters of the following masses 1.0, 1.75, 2.25.

is in rest. Dividing the first two of Eq. (17) by F_{11} and the other two by F_{31} one gets

$$\begin{aligned}
&m_1 + m_2 \cos\beta_{\rm II} + m_3 \cos\beta_{\rm III} + m_4 \cos\beta_{\rm IV} + m_5 \cos\beta_{\rm V} = 0,\\
&m_2 \sin\beta_{\rm II} - m_3 \sin\beta_{\rm III} + m_4 \sin\beta_{\rm IV} - m_5 \sin\beta_{\rm V} = 0,\\
&m_1 + m_2 \cos 3\beta_{\rm II} + m_3 \cos 3\beta_{\rm III} + m_4 \cos 3\beta_{\rm IV} + m_5 \cos 3\beta_{\rm V} = 0,\\
&m_2 \sin 3\beta_{\rm II} - m_3 \sin 3\beta_{\rm III} + m_4 \sin 3\beta_{\rm IV} - m_5 \sin 3\beta_{\rm V} = 0.
\end{aligned} \tag{18}$$

Equation (18) has been solved by the method of searching for the zero minimum of

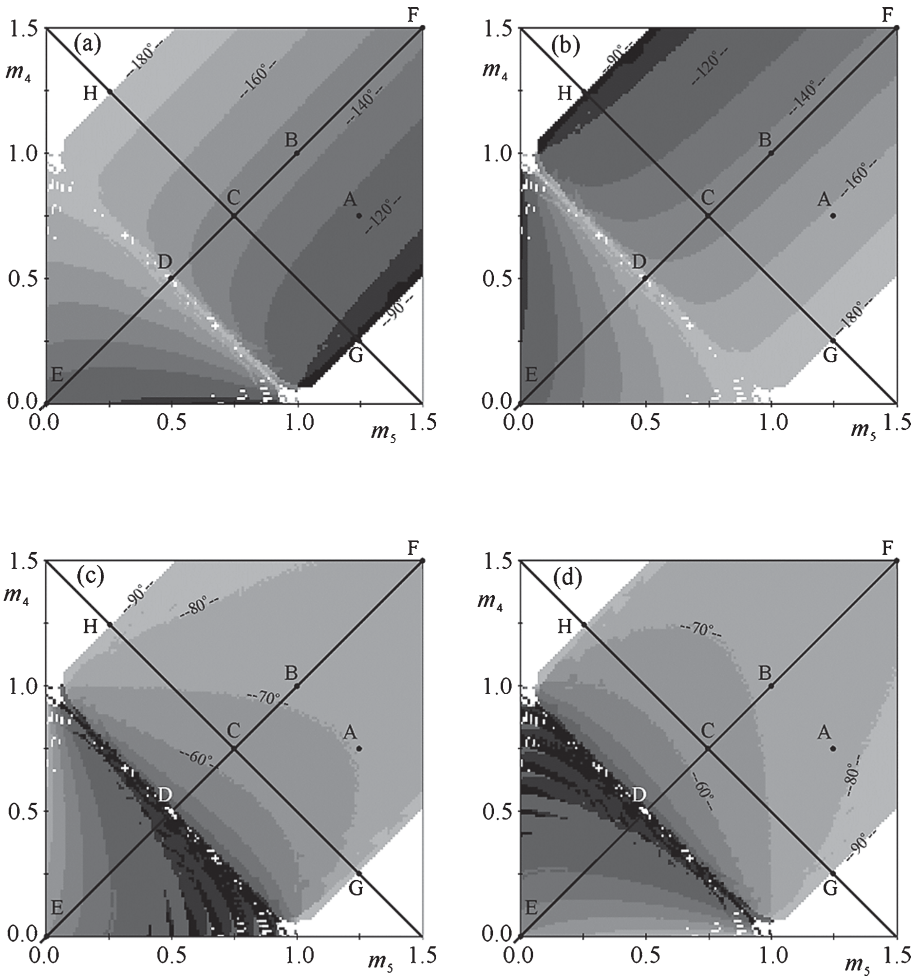

Fig. 9. Contour maps of phase differences (a) $\beta_{\rm II}$, (b) $\beta_{\rm III}$, (c) $\beta_{\rm IV}$, (d) $\beta_{\rm V}$, versus pendula masses m_4 and m_5; $m_1 = m_2 = m_3 = 1.0$.

the functions H_{15} and H_{35}:

$$\begin{aligned} H_{15} &= (m_1 + m_2\cos\beta_{\rm II} + m_3\cos\beta_{\rm III} + m_4\cos\beta_{\rm IV} + m_5\cos\beta_{\rm V})^2 \\ &\quad + (m_2\sin\beta_{\rm II} - m_3\sin\beta_{\rm III} + m_4\sin\beta_{\rm IV} - m_5\sin\beta_{\rm V})^2\,, \\ H_{35} &= (m_1 + m_2\cos 3\beta_{\rm II} + m_3\cos 3\beta_{\rm III} + m_4\cos 3\beta_{\rm IV} + m_5\cos 3\beta_{\rm V})^2 \\ &\quad + (m_2\sin 3\beta_{\rm II} - m_3\sin 3\beta_{\rm III} + m_4\sin 3\beta_{\rm IV} - m_5\sin 3\beta_{\rm V})^2\,. \end{aligned} \tag{19}$$

The functions H_{15} and H_{35} represent respectively the square of the amplitude of the first and third harmonic components of the force acting on the beam M. Notice that the phase differences $\beta_{\rm II}$, $\beta_{\rm III}$, $\beta_{\rm IV}$ and $\beta_{\rm V}$ calculated from Eq. (18) do not depend on the models of escapement mechanism and friction.

The dependence of the phase differences $\beta_{\rm II}$, $\beta_{\rm III}$, $\beta_{\rm IV}$ and $\beta_{\rm V}$, i.e., the values for which functions H_{15} and H_{35} have zero minimum, on the parameters m_1, m_2, m_3, m_4 and m_5 is partially described in Figs. 9(a)–(d). To reduce the dimensionality and allow the visualization we consider $m_1 = m_2 = m_3 = 1.0$ (three identical pendula), and allow m_4 and m_5 to vary in the interval [0.0, 1.5]. The contour maps of β_{II}, β_{III}, β_{IV} and β_V are shown respectively in Figs. 9(a)–(d). The phase differences β_{II} and β_{III} are shown in the interval [90°, 180°] and β_{IV}, β_V in the interval [0°, 90°]. In Figs. 9(a)–(d) a few characteristic points are indicated. Point A ($m_4 = 0.75$, $m_5 = 1.25$) represents the phase synchronization of five pendula: $m_1 = m_2 = m_3 = 1.0$, $m_4 = 0.75$, $m_5 = 1.25$, with phase differences given by: $\beta_{II} = 124.5°$, $\beta_{III} = 162°$, $\beta_{IV} = 71°$, $\beta_V = 77°$ (this configuration is shown in Fig. 8(a)). On the line $m_4 = m_5$ the symmetrical configurations are located. The symmetrical configuration of five identical clocks is indicated by point B. Point C ($m_5 = m_4 = 0.75$) is characteristic for the configuration for $m_1 = m_2 = m_3 = 1.0$ and $m_4 = m_5 = 0.75$ with phase differences $\beta_{II} = \beta_{III} = 145.5°$ and $\beta_{IV} = \beta_V = 65.5°$. The configuration of the point D ($m_5 = m_4 = 0.5$) is shown in Fig. 10(a). The phase differences for pendula 2 and 3 ($m_2 = m_3 = 1.0$) are equal to $\beta_{II} = \beta_{III} = 180°$, and for pendula 4 and 5 $\beta_{IV} = \beta_V = 0°$. It is a limit case in which two clusters of two and three pendula, but with the same mass $m_1 + m_4 + m_5 = m_2 + m_3 = 2.0$, are in antiphase to each other. Point E represents the system with $m_5 = m_4 = 0.0$, i.e., the system with three clocks, $m_1 = m_2 = m_3 = 1.0$ and phase differences $\beta_{II} = \beta_{III} = 120°$. Point F ($m_5 = m_4 = 1.5$) describes the configuration for $m_2 = m_3 = 1.0$, $m_4 = m_5 = 1.5$ and phase differences $\beta_{II} = \beta_{III} = 143°$ and $\beta_{IV} = \beta_V = 78°$. Observe that with the further increase of the masses of pendula 4 and 5, i.e., for $m_5 = m_4 \to \infty$, phase differences $\beta_{IV} = \beta_V \to 90°$, so pendula 4 and 5 oscillate in antiphase, and phase differences $\beta_{II} = \beta_{III} \to 120°$. In this case we have the co-existence of two configurations; two large pendula with equal masses $m_5 = m_4$ oscillate in antiphase and other pendula with masses $m_1 = m_2 = m_3 = 1.0$ are phase synchronized with phase differences $\beta_{II} = \beta_{III} = 120°$ as shown in Fig. 10(b). Point G is the crossing point of the lines $m_4 = 1.5 - m_5$ and $m_4 = m_5 - 1$. It represents the system with $m_4 = 0.25$ and $m_5 = 1.25$, in which phase difference of pendulum 3 ($m_3 = 1.0$) is equal to $\beta_{III} = 180°$ and this pendulum is in antiphase to pendulum 1 ($m_1 = 1$). Phase differences of pendula 2 ($m_2 = 1.0$) and 4 ($m_4 = 0.25$) are $\beta_{II} = \beta_{IV} = 90°$, which means that these pendula create a cluster which is in antiphase to pendulum 5 ($m_5 = 1.0$) for which $\beta_V = 90°$. This is a special case when there are four clusters with antiphase synchronization in pairs – see Fig. 8(c). The symmetrical configuration to the one described in Fig. 10(c) is shown in Fig. 10(d) and represented by point H in Figs. 9(a)–(d).

Similar contour maps can be obtained for different intervals of phase differences β_{II}, β_{III}, β_{IV} and β_V. For some values of the parameters m_4 and m_5 more than one phase synchronization configurations of five clusters exist. i.e., for some values of m_4 and m_5 there exist more than one different set of phase differences β_{II}, β_{III}, β_{IV} and β_V for which functions H_{15} and H_{35} have zero minima.

The method of the phase shifts estimation derived for five pendula (Eqs. (17)–(19)) can be generalized to any number of pendula synchronized in five clusters. Substituting the total masses of clusters $\bar{m}_{1-5}$ instead of pendulum masses m_{1-5} one gets equations which allow estimation of the phase shifts between clusters.

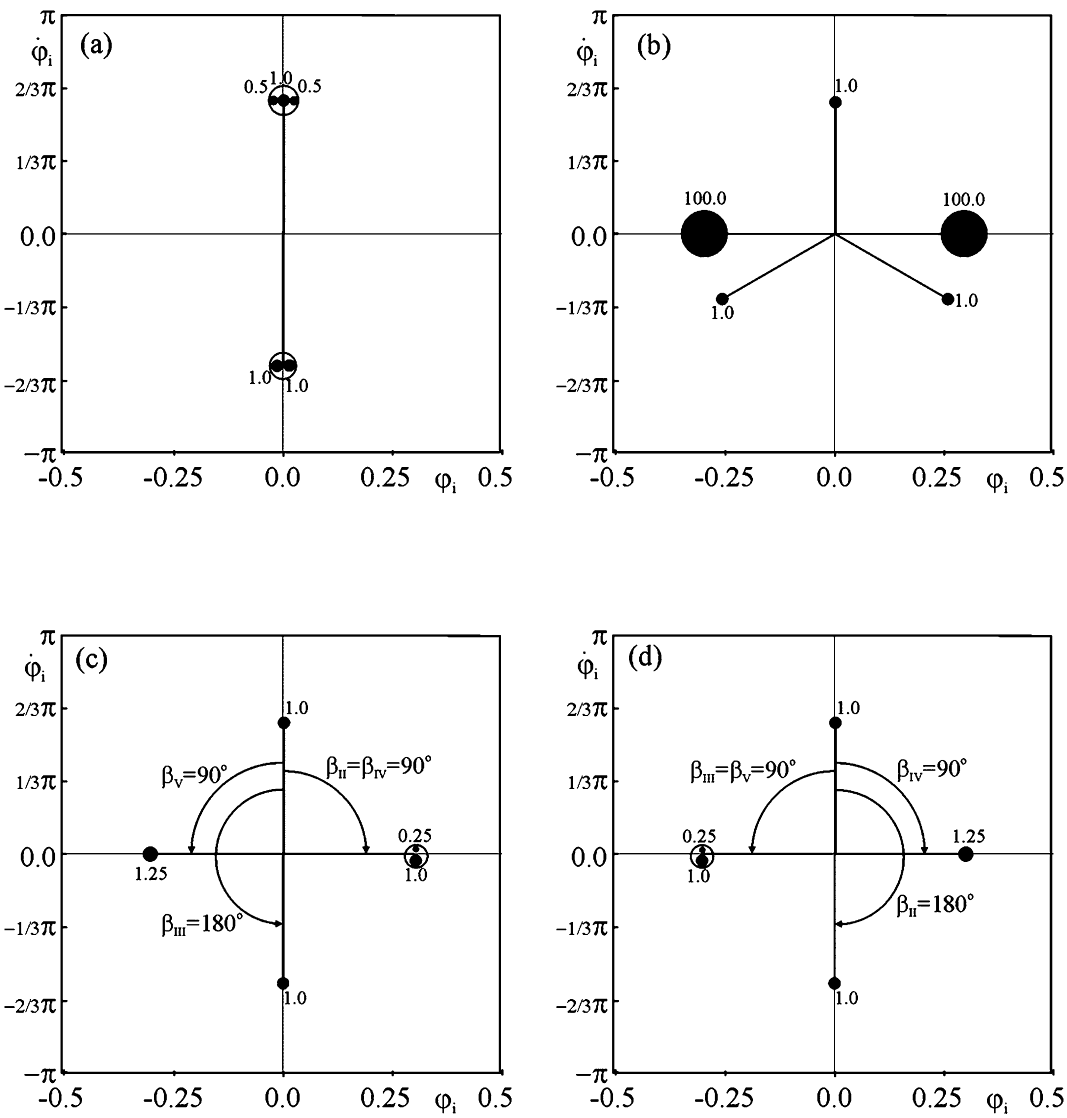

Fig. 10. Synchronization configurations of five pendula; (a) antiphase configuration of two clusters of the mass 2.0 each, (b) coexistence of two heavy pendula in antiphase with the configuration of three pendula mass 1.0 each, (c) coexistence of two pairs of two clusters in antiphase; (d) mirror image of previous configuration.

§4. Discussion and conclusions

For some values of the parameters m_4 and m_5 there exist more than one phase synchronization configurations of five clusters, i.e., for m_4 and m_5 there exist more than one different set of phase differences II, III, IV and V for which functions H15 and H35 have zero minima. To explain why other cluster configurations are not possible goes back to the approximation given by Eq. (6). The pendula act on the beam M with the force which consists only of the first and third harmonics

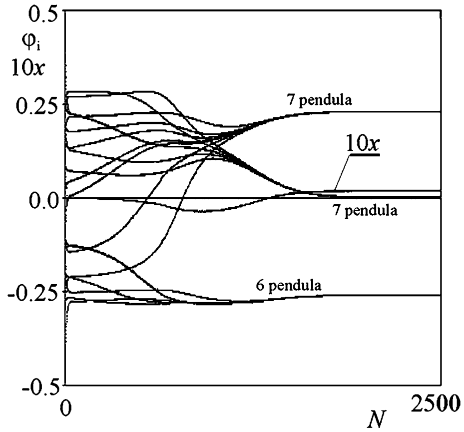

Fig. 11. Synchronization of 20 clocks: $m_i = 1.0$, $l_i = 0.2485$, $M = 10.0$, $k_x = 118.4$, $c_x = 24.0$. N on the parallel axis indicated the number of periods of the pendula oscillations.

of the pendula's oscillations frequency α. Three clusters configuration occurs when the first harmonic of this force is equal to zero and the system tends to five clusters configuration when both harmonics are equal to zero. This result is exactly the same as in the case of identical clocks (for details see Refs. 13) and 14)) and is general for the problems of clocks' synchronization but contrary to the case of identical clocks[13),14)] clustering in three and five clusters is easily observable for both even and odd numbers of clocks. For an even number of clocks the creation of the pairs of clocks synchronized in antiphase is possible only in the special non-robust case of two groups of identical clocks. Due to the assumption (3) and the small swings of the pendula ($\Phi < 2\pi/36$) other harmonics do not exist (or are extremely small). One can expect the appearance of other numbers of clusters, when the pendula's swings are larger and their periodic oscillations will be described by higher harmonic components, but this is not the case of the pendula clocks.

We studied the systems with up to 100 clock. It has been found that for larger n, randomly distributed differences of pendula masses three clusters configurations are more probable than five clusters ones. As an example consider the case of 20 clocks shown in Fig. 11. After the initial transient pendula with randomly distributed masses $m_i = 1.0 \pm 0.1$ create three clusters with respectively 6, 7 and 7 pendula. Notice that as described in §3.2 the beam is oscillating with a small amplitude.

As in the case of identical clocks[13),14)] we show that the clocks clustering phenomena take place far below the resonances for both longitudinal and transverse oscillations of the beam so the influence of these oscillations can be neglected.

To summarize, we have studied the phenomenon of the synchronization in the array of non-identical pendulum clocks hanging from an elastically fixed horizontal beam. We show that besides the complete synchronization of all pendulum clocks, the pendula can be grouped either in three or five clusters only. The pendula in the clusters perform complete synchronization and the clusters are in the form of the phase synchronization characterized by a constant phase difference between the pendula given by Eq. (13) for three pendula and Eq. (17) for five pendula. All the pendula configurations reported in this paper are stable and robust as they exist for the given sets of system (1) and (2) parameters which have positive Lebesque measure.

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