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Synchronization of the self-excited pendula suspended on the vertically displacing beam

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ABSTRACT

We consider the synchronization of a number of self-excited identical pendula hanging on the same beam which can move vertically. We identify different synchronous configurations and investigate their stability. An approximate analytical analysis of the energy balance allows to derive the synchronization conditions, phase difference between pendula and explains the observed types of synchronizations.

1. Introduction

Recent investigations have shown that the coupled systems have great potential in a large amount of application areas ranging from physics and engineering to economy, biology and medicine [1,3,15]. The main interest in these studies is focused on the phenomenon of synchronization as the phenomenon of mutual synchronization offers the most fundamental example of emergent behavior. It is the process where two or more systems interact with each other and come to oscillate together. Groups of oscillators are observed to synchronize in a diverse variety of systems, despite the inevitable differences between the oscillators. Synchronization is widespread in nature and common in mechanical oscillatory systems.

The phenomenon of the synchronization of the clocks hanging on a common movable beam [10] has been recently the subject of research by a number of authors [2,4–9,11–14,16–19]. These studies give the definite answer to the question; what Huygens was able to observe, e.g., Bennet et al. [2] state that to repeat Huygens' results, the high precision (the precision that Huygens certainly could not achieve) is necessary and Kanunnikov and Lamper [11] show that the precise antiphase motion of different pendula noted by Huygens cannot occur. Our studies [4–7] prove that in the case of nonidentical clocks only almost antiphase synchronization can be observed.

In the previous papers [4,5,10] we studied a synchronization problem for n pendulum clocks hanging from an elastically fixed horizontal beam. It was assumed that each pendulum performs a periodic motion which starts from different initial conditions. We showed that after a transient different types of synchronization between pendula can be observed. The first type is in-phase complete synchronization in which all pendula behave identically. In the second type one can identify the groups (clusters) of synchronized pendula. We showed that only configurations of three and five clusters are possible and derive algebraic equations for the phase difference between the pendula in different clusters.

In this paper we consider the case of n identical self-excited pendula hanging from the same beam. The oscillations of each pendulum are self-excited by van der Pol's type of damping. Contrary to the previously considered cases we assume

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that the beam can oscillate not in the horizontal but in vertical direction. The possible synchronous configurations have been identified and their stability has been investigated. We consider the energy balance in the system, derive the synchronization conditions and explains the observed types of synchronizations. We argue that our results are robust as they exist in the wide range of system parameters.

This paper is organized as follows. Section 2 describes the considered model of the coupled pendula. In Section 3 we derive the energy balance of the synchronized pendula. Section 4 presents the results of our numerical simulations and describes the observed synchronization states. Finally, we summarize our results in Section 5.

2. Model

The analyzed system is shown in Fig. 1. It consists of a rigid beam and a number of pendula suspended on it. The beam of mass M can displace only vertically (it has one degree of freedom); it is connected with the frame by a lightweight linear spring of the stiffness coefficient K_Y and a viscous damper of the damping coefficient C_Y. The displacement of the beam is described by coordinate Y. The pendula have the form of mathematical pendula of lengths L_i and masses M_i. The motion of the pendula is described by angles φ_i and is self-excited by van der Pol's type of damping (not shown in Fig. 1) given by momentum (torque), $C_{\varphi_i}\frac{d\varphi_i}{dt(1-C_{VDPi}\varphi_i^2)}$ where C_{φ_i} and C_{VDPi} are constant.

The equation of motion for the above-described system can be written as follows:

$$M_iL_i^2\frac{d^2\varphi_i}{dt^2} - M_iL_i\frac{d^2Y}{dt^2}\sin\varphi_i + C_{\varphi i}(1.0 - C_{VDPi}\varphi_i^2)\frac{d\varphi_i}{dt} + M_iL_ig\sin\varphi_i = 0.0, \quad i = 1\ldots n \tag{1}$$

$$\left(M_B + \sum_{i=1}^{n} M_i\right)\frac{d^2Y}{dt^2} + C_Y\frac{dY}{dt} + K_YY = \sum_{i=1}^{n} M_iL_i\left(\frac{d^2\varphi_i}{dt^2}\sin\varphi_i + \left(\frac{d\varphi_i}{dt}\right)^2\cos\varphi_i\right) \tag{2}$$

Considering mass M_1 and length L_1 of the first pendulum and the gravitational acceleration g as the reference quantities, the dimensional Eqs. (1,2) can be rewritten in a dimensionless form as:

$$m_il_i^2\ddot{\varphi}_i - m_il_i\ddot{y}\sin\varphi_i + c_{\varphi i}(1.0 - c_{VDPi}\varphi_i^2)\dot{\varphi}_i + m_il_i\sin\varphi_i = 0.0, \quad i = 1\ldots n \tag{3}$$

$$\left(m_B + \sum_{i=1}^{n} m_i\right)\ddot{y} + c_y\dot{y} + k_yy = \sum_{i=1}^{n} m_il_i(\ddot{\varphi}_i\sin\varphi_i + \dot{\varphi}_i^2\cos\varphi_i) \tag{4}$$

The relationships between the dimensional quantities of Eqs. (1) and (2) and the dimensionless quantities of Eqs. (3) and (4) are as follow: $m_i = \frac{M_i}{M_1}$ (dimensionless mass of the ith pendulum), $l_i = \frac{L_i}{L_1}$ (dimensionless length of the ith pendulum), $\tau = \alpha t$ (dimensionless time), $\alpha = \sqrt{\frac{g}{L_1}}$, $y = \frac{Y}{L_1}$ (dimensionless displacement of the beam M), $c_{\varphi i} = \frac{C_{\varphi i}\sqrt{L_1}}{M_1L_1^2\sqrt{g}}$, $c_{VDPi} = C_{VDPi}$, $m_B = \frac{M_B}{M_1}$ $c_y = \frac{C_Y\sqrt{L_1}}{M_1\sqrt{g}}$, $k_y = \frac{K_YL_1}{M_1g}$, symbols $\ddot{\ }$ and $\dot{\ }$ denote respectively $\frac{d^2}{d\tau^2}$ and $\frac{d}{d\tau}$.

In the case of a system composed of identical pendula, Eqs. (3) and (4) take the form:

$$\ddot{\varphi}_i - \ddot{y}\sin\varphi_i + c_{\varphi i}(1.0 - c_{VDPi}\varphi_i^2)\dot{\varphi}_i + \sin\varphi_i = 0.0, \quad i = 1\ldots n \tag{5}$$

$$(m_B + n)\ddot{y} + c_y\dot{y} + k_yy = \sum_{i=1}^{n}(\ddot{\varphi}_i\sin\varphi_i + \dot{\varphi}_i^2\cos\varphi_i) \tag{6}$$

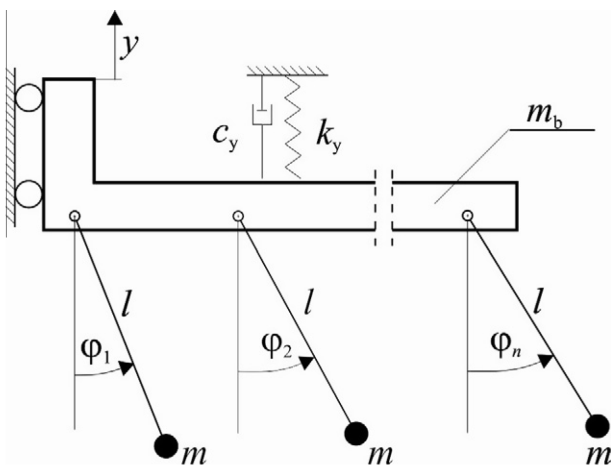

Fig. 1. Pendula suspended on the vertically movable beam.

One can say that the state of synchronization is attained when the motion of the system is periodic and this takes place when the work (denoted below as W^{SYN}) performed during one period of the pendulum motion by the force with which the pendulum acts on the beam is equal to zero, then the pendulum energy remains constant.

Let us multiply Eq. (3) by the pendulum velocity. We will arrive at the equation of the power balance in the form:

$$m_i l_i^2 \ddot{\varphi}_i \dot{\varphi}_i + m_i l_i \dot{\varphi}_i \sin \varphi_i = -c_{\varphi i} \dot{\varphi}_i^2 + c_{\varphi i} c_{VDPi} \varphi_i^2 \dot{\varphi}_i^2 + m_i l_i \ddot{y} \sin \varphi_i \dot{\varphi}_i, \quad i = 1 \ldots n \tag{7}$$

When the pendula and the whole system move periodically, the integration with respect to time of Eq. (7) takes the form of equations of energy balance:

$$\int_0^T m_i l_i^2 \ddot{\varphi}_i \dot{\varphi}_i d\tau + \int_0^T m_i l_i \dot{\varphi}_i \sin \varphi_i d\tau == -\int_0^T c_{\varphi i} \dot{\varphi}_i^2 d\tau + \int_0^T c_{\varphi i} c_{VDPi} \varphi_i^2 \dot{\varphi}_i^2 d\tau + \int_0^T m_i l_i \ddot{y} \sin \varphi_i \dot{\varphi}_i d\tau, \quad i = 1 \ldots n \tag{8}$$

The left-hand side of Eq. (8) denotes an increment in the total energy of the pendulum. When the pendulum together with the whole system move periodically, this increment is equal to zero:

$$\int_0^T m_i l_i^2 \ddot{\varphi}_i \dot{\varphi}_i d\tau + \int_0^T m_i l_i \dot{\varphi}_i \sin \varphi_i d\tau = 0, \quad i = 1, \ldots, n \tag{9}$$

The first term on the right-hand side of Eq. (8) denotes the energy supplied during one period of oscillations by the van der Pol's damper:

$$W_i^{DAMP} = -\int_0^T c_{\varphi i} \dot{\varphi}_i^2 d\tau, \quad i = 1, \ldots n \tag{10}$$

The next term refers to the energy dispersed during one period of oscillations by the van der Pol's damper:

$$W_i^{VDP} = \int_0^T c_{\varphi i} c_{VDPi} \varphi_i^2 \dot{\varphi}_i^2 dt, \quad i = 1, \ldots, n \tag{11}$$

The last term on the right-hand side of Eq. (8) represents the energy transferred by the pendulum to the moving beam (the energy lost by the pendulum due to the beam motion):

$$W_i^{SYN} = \int_0^T m_i l_i \ddot{y} \sin \varphi_i \dot{\varphi}_i d\tau, \quad i = 1, \ldots, n \tag{12}$$

Substituting Eqs. (9)–(12) into Eq. (8), we obtain energy balances of pendula in the form:

$$W_i^{DAMP} + W_i^{VDP} + W_i^{SYN} = 0, \quad i = 1, \ldots, n \tag{13}$$

Equaling to zero the work of synchronization expressed by Eq. (12) does not lead to any effective conclusions, thus the next step in the analysis of the synchronization phenomenon is an assumption of the harmonic character of the pendulum motion. If the amplitude of the beam motion and the amplitude of the pendulum motion are low enough, it can be assumed that the pendulum motion is described by the harmonic function:

$$\varphi_i = \Phi_i \sin(\tau + \beta_i) \tag{14}$$

(note that the frequency of pendulum free oscillations equals to 1 in the dimensionless notation), and thus the functions describing velocity and acceleration are as follow:

$$\begin{aligned} \dot{\varphi}_i &= \Phi_i \cos(\tau + \beta_i) \\ \ddot{\varphi}_i &= -\Phi_i \sin(\tau + \beta_i) \end{aligned} \tag{15}$$

After the substitution of Eqs. (14) and (15) into beam motion Eq. (4), one obtains:

$$\left(m_b + \sum_{i=1}^n m_i\right)\ddot{y} + c_y \dot{y} + k_y y = \sum_{i=1}^n \left(-m_i l_i \Phi_i^2 \sin^2(\tau + \beta_i) + m_i l_i \Phi_i^2 \cos^2(\tau + \beta_i)\right) \tag{16}$$

Considering that $\cos^2 \alpha - \sin^2 \alpha = \cos 2\alpha$, and denoting:

$$U = m_b^+ \sum_{i=1}^n m_i \tag{17}$$

one gets:

$$U\ddot{y} + c_y \dot{y} + k_y y = \sum_{i=1}^n \left(m_i l_i \Phi_i^2 \cos(2\tau + 2\beta_i)\right) \tag{18}$$

For a sufficiently small value of damping coefficient c_y, the solution of Eq. (18) takes the following form

$$y = \sum_{i=1}^{n} (Y_i \cos(2\tau + 2\beta_i)) \tag{19}$$

where:

$$Y_i = \frac{m_i l_i \Phi_i^2}{k_y - 4U} \tag{20}$$

Eqs. (19) and (20) allow to derive the beam acceleration:

$$\ddot{y} = \sum_{i=1}^{n} (A_i \cos(2\tau + 2\beta_i)) \tag{21}$$

where:

$$A_i = -\frac{4 m_i l_i \Phi_i^2}{k_x - 4U} \tag{22}$$

The synchronization state of the pendulum motion is in other words the periodic motion of the system in which phase angles β_I are constant. In this state, when pendula are identical there is no energy transfer from one pendulum *via* the beam to another pendulum. In the process of synchronization, such a state in which the synchronization work (12) during one period of motion is equal to zero, is aimed at. Substituting Eqs. (14), (15), (21) into Eq. (12), and equaling Eq. (12) to zero, we obtain:

$$W_i^{SYN} = \int_0^T m_i l_i \left(\sum_{k=1}^{n} (A_k \cos(2\tau + 2\beta_k)) \right) \Phi_i^2 \sin(\tau + \beta_i) \cos(\tau + \beta_i) d\tau = 0 \tag{23}$$

After further transformations, we arrive at:

$$W_i^{SYN} = m_i l_i \Phi_i^2 \int_0^T \left(\sum_{k=1}^{n} (A_k \cos(2\tau + 2\beta_k)) \right) 0.5 \sin(2\tau + 2\beta_i) d\tau \tag{24}$$

$$W_i^{SYN} = 0.5 m_i l_i \Phi_i^2 \left(\sum_{k=1}^{n} A_k \int_0^T ((\cos 2\tau \cos 2\beta_k - \sin 2\tau \sin 2\beta_k)(\sin 2\tau \cos 2\beta_i + \cos 2\tau \sin 2\beta_i)) d\tau \right) = 0 \tag{25}$$

$$W_i^{SYN} = 0.5 m_i l_i \Phi_i^2 \sum_{k=1}^{n} A_k \begin{pmatrix} \cos 2\beta_k \cos 2\beta_i \int_0^T \cos 2\tau \sin 2\tau d\tau \\ + \cos 2\beta_k \sin 2\beta_i \int_0^T \cos 2\tau \cos 2\tau d\tau \\ - \sin 2\beta_k \cos 2\beta_i \int_0^T \sin 2\tau \sin 2\tau d\tau \\ - \sin 2\beta_k \sin 2\beta_i \int_0^T \sin 2\tau \cos 2\tau d\tau \end{pmatrix} = 0 \tag{26}$$

As:

$$\begin{aligned} &\int_0^T \cos 2\tau \sin 2\tau d\tau = 0.0 \\ &\int_0^T \sin 2\tau \sin 2\tau d\tau = 0.5T = \pi \\ &\int_0^T \cos 2\tau \cos 2\tau d\tau = 0.5T = \pi \end{aligned} \tag{27}$$

Eq. (26) takes the form:

$$\begin{aligned} W_i^{SYN} &= 0.5 m_i l_i \Phi_i^2 \sum_{k=1}^{n} A_k \pi (\cos 2\beta_k \sin 2\beta_i - \sin 2\beta_k \cos 2\beta_i) = 0.5 \pi m_i l_i \Phi_i^2 \sum_{k=1}^{n} A_k \sin(2\beta_i - 2\beta_k) \\ &= \frac{-2\pi m_i l_i \Phi_i^2}{k_x - 4U} \sum_{k=1}^{n} m_k l_k \Phi_k^2 \sin(2\beta_i - 2\beta_k) = 0.0. \end{aligned} \tag{28}$$

Eq. (28) is satisfied for each pendulum (i.e., for each $i = 1, \ldots, n$) by two groups of phase angles. The first group fulfills the condition:

$$\sum_{k=1}^{n} m_k l_k \Phi_k^2 \sin(2\beta_i - 2\beta_k) = 0.0, \quad i = 1, \ldots n \tag{29}$$

as it satisfies the equations:

$$\sin(2\beta_i - 2\beta_k) = 0.0, \quad i = 1, \ldots n, \; k = 1, \ldots, n \tag{30}$$

If $\beta_1 = 0°$ is assumed (as one of the phase angles can be chosen arbitrarily), then condition (30) is satisfied evidently by all combinations of the angles $\beta_{1,2,\ldots,n} = 0°$, 90°, 180° and 270°. The second group of the phase angles corresponding to synchronous configurations can be determined after the following transformation of Eq. (28):

$$\begin{aligned}\sum\nolimits_{k=1}^{n} m_k l_k \Phi_k^2 \sin(2\beta_i - 2\beta_k) &= \sum\nolimits_{k=1}^{n} m_k l_k \Phi_k^2 (\sin 2\beta_i \cos 2\beta_k - \cos 2\beta_i \sin 2\beta_k) \\ &= \sin 2\beta_i \sum\nolimits_{k=1}^{n} m_k l_k \Phi_k^2 (\cos 2\beta_k) - \cos 2\beta_i \sum\nolimits_{k=1}^{n} m_k l_k \Phi_k^2 (\sin \beta_k) = 0.0 \\ &\Rightarrow \sum\nolimits_{k=1}^{n} m_k l_k \Phi_k^2 (\cos 2\beta_k) = 0.0 \wedge \sum\nolimits_{i=1}^{n} m_k l_k \Phi_k^2 (\sin 2\beta_k) = 0.0\end{aligned} \tag{31}$$

Eq. (31) in the case of identical masses of pendula and identical amplitudes of oscillations is simplified to the form:

$$\sum_{i=1}^{n} \cos 2\beta_i = 0, \quad \sum_{i=1}^{n} \sin 2\beta_i = 0 \tag{32}$$

3. Examples of pendula configurations

3.1. Two pendula

When $n = 2$ and $\beta_1 = 0.0°$, condition (30) is satisfied by the following phase angles:

(i) complete synchronization:

$$\begin{aligned}\beta_1 &= 0^{\circ} \\ \beta_2 &= 0^{\circ}\end{aligned} \tag{33}$$

(ii) antiphase synchronization:

$$\begin{aligned}\beta_1 &= 0^{\circ} \\ \beta_2 &= 180^{\circ}\end{aligned} \tag{34}$$

(iii) quarter-phase synchronization:

$$\begin{aligned}\beta_1 &= 0^{\circ} \\ \beta_2 &= 90^{\circ}\end{aligned} \tag{35}$$

(we do not distinguish the configurations of $\beta_1 = 0°$ and $\beta_2 = 270°$, qualitatively identical to the above-mentioned quarter-phase synchronization, as a separate case).

The quarter-phase configuration for which $\beta_2 = 90°$ fulfills additionally the condition defining the second group of angles, expressed by Eq. (32), which for two pendula takes the form:

$$\begin{aligned}1.0 + \cos 2\beta_2 &= 0 \\ \sin 2\beta_2 &= 0\end{aligned} \tag{36}$$

3.2. Three pendula

When $n = 3$ and $\beta_1 = 0°$, condition (30) takes the form:

$$\begin{aligned}\sin(2\beta_1 - 2\beta_2) &= 0 \\ \sin(2\beta_1 - 2\beta_3) &= 0 \\ \sin(2\beta_2 - 2\beta_3) &= 0\end{aligned} \tag{37}$$

and is satisfied by the following phase angles:

(i) complete synchronization:

$$\begin{aligned}\beta_1 &= 0^{\circ} \\ \beta_2 &= 0^{\circ} \\ \beta_3 &= 0^{\circ}\end{aligned} \tag{38}$$

(ii) antiphase synchronization of two clusters composed of one pendulum and two pendula:

$$\beta_1 = 0^\circ$$
$$\beta_2 = 180^\circ$$
$$\beta_3 = 180^\circ \tag{39}$$

(as in Section 3.1, we do not distinguish the configurations of $\beta_1 = 0°$, $\beta_2 = 0°$, $\beta_3 = 180°$ and $\beta_1 = 0°$, $\beta_2 = 180°$, $\beta_3 = 0°$ qualitatively identical to the above-mentioned antiphase synchronization, as separate cases)

(iii) quarter-phase synchronization of two clusters composed of one pendulum and two pendula (unstable – the stability investigation: see below):

$$\beta_1 = 0^\circ$$
$$\beta_2 = 90^\circ$$
$$\beta_3 = 90^\circ \tag{40}$$

(iv) quarter-phase synchronization of three clusters composed of one pendulum each (also unstable):

$$\beta_1 = 0^\circ$$
$$\beta_2 = 90^\circ$$
$$\beta_3 = 180^\circ \tag{41}$$

For the second group of angles, condition (32) takes the form:

$$\begin{aligned} &1.0 + \cos 2\beta_2 + \cos 2\beta_3 = 0 \\ &\sin 2\beta_2 + \sin 2\beta_3 = 0 \end{aligned} \tag{42}$$

and is not satisfied by the above-mentioned phase angles. However, the condition is fulfilled by the angles:

(v) 60° – phase synchronization:

$$\beta_1 = 0.0^\circ, \quad \beta_2 = 60.0^\circ, \quad \beta_3 = 120.0^\circ \tag{43}$$

(vi) 120° – phase synchronization:

$$\beta_1 = 0.0^\circ, \quad \beta_2 = 120.0^\circ, \quad \beta_3 = 240.0^\circ \tag{44}$$

3.3. Four pendula

For systems with four pendula, the authors distinguish the following kinds of synchronization for the phase angles satisfying condition (30): (i) complete synchronization, when $\beta_{1,2,\dots,4} = 0°$ – all pendula form one cluster, (ii) antiphase synchronization of two clusters composed of k and 4-k pendula: $\beta_{1,2,\dots,k} = 0°$, $\beta_{k+1,\dots,4} = 180°$, where k can have values from 1 to 3 depending on the initial conditions. The observed here size of configurations results from the fact that the function describing the resultant force the pendula exert on the beam does not depend on the number of pendula in each cluster; it is the same as in the case of the complete synchronization.

For four pendula, condition (32) will take the form:

$$\begin{aligned} &1.0 + \cos 2\beta_2 + \cos 2\beta_3 + \cos 2\beta_4 = 0 \\ &\sin 2\beta_2 + \sin 2\beta_3 + \sin 2\beta_4 = 0 \end{aligned} \tag{45}$$

As can be seen, it is a system of two equations with three unknowns, thus the value of one of the phase angles, for instance of β_3, can be assumed arbitrarily (again, it depends on the initial conditions). It turns out that for $\beta_1 = 0°$ and an arbitrary value of β_3, the system of Eq. (45) has the following solution:

$$\beta_1 = 0.0^\circ, \quad \beta_2 = 90.0^\circ, \quad \beta_3 = \beta_3, \quad \beta_4 = \beta_3 + 90.0^\circ \tag{46}$$

It means that the synchronous configuration (called 90°+90° synchronization) consists of two pairs of in the quarter-phase synchronization (observed in the systems with two pendula); the phase angle between these pairs depends on the initial conditions and can be subjected to permanent alternations due to disturbances.

3.4. Beam motion in various states of synchronization

The right-hand side of Eq. (18) represents the total (resultant) force F the pendula exert on the beam:

$$F = \sum_{i=1}^{n} \left(m_i l_i \Phi_i^2 \cos(2\tau + 2\beta_i) \right) \tag{47}$$

Let us transform Eq. (47) to the form:

$$F=\sum_{i=1}^{n}\left(m_il_i\Phi_i^2\cos(2\tau+2\beta_i)\right)=m_1l_1\Phi_1^2\sum_{i=1}^{n}(\cos2\tau\cos2\beta_i-\sin2\tau\sin2\beta_i)=$$
$$=\cos2\tau\left(m_1l_1\Phi_1^2\sum_{i=1}^{n}(\cos2\beta_i)\right)-\sin2\tau\left(m_1l_1\Phi_1^2\sum_{i=1}^{n}(\sin2\beta_i)\right)=0.0 \tag{48}$$

It can be easily seen that for the phase angles fulfilling condition (32), the total force the pendula act on the beam is equal to zero and the beam does not displace. It takes place for the quarter-phase synchronization of two pendula, the 60°-synchronization and the 120°-synchronization of three pendula and the 90°+90° synchronization of four pendula. In the remaining states of synchronization, the forces the pendula act on the beam do not balance out. Under the action of the resultant force of the frequency 2, the beam performs harmonic oscillations of this frequency, twice as high as the frequency of the oscillating motion of the pendula.

3.5. Stability of synchronous configurations

Having found which phase angles fulfill synchronization conditions (30) and (or) (32), it is indispensable to investigate the stability of these configurations. The equations of disturbances corresponding to Eqs. (3) and (4) are as follow:

$$m_il_i^2\ddot{\xi}_i-m_il_i\ddot{z}\sin\varphi_i=-m_il_i(1-\ddot{y})\xi_i\cos\varphi_i-c_{\varphi i}\left(1.0-c_{VDPi}\varphi_i^2\right)\dot{\xi}_i i=1\dots n \tag{49}$$

$$\left(m_B+\sum_{i=1}^{n}m_i\right)\ddot{z}-\sum_{i=1}^{n}m_il_i\ddot{\xi}_i\sin\varphi_i=-c_y\dot{z}-k_yz+\sum_{i=1}^{n}m_il_i\left(\ddot{\varphi}_i\xi_i\cos\varphi_i-\dot{\varphi}_i^2\xi_i\sin\varphi_i+2\dot{\varphi}_i\dot{\xi}_i\cos\varphi_i\right) \tag{50}$$

where ξ_I are disturbances of the displacements φ_i and z is a disturbance of the beam displacement y. In the case of the system with identical pendula, the equations of disturbances corresponding to equations of motion (5) and (6) take the form:

$$\ddot{\xi}_i-\ddot{z}\sin\varphi_i=-(1-\ddot{y})\xi_i\cos\varphi_i-c_{\varphi i}\left(1.0-c_{VDPi}\varphi_i^2\right)\dot{\xi}_i \tag{51}$$

$$(m_B+n)\ddot{z}-\sum_{i=1}^{n}\ddot{\xi}_i\sin\varphi_i==-c_y\dot{z}-k_yz+\sum_{i=1}^{n}\left(\ddot{\varphi}_i\xi_i\cos\varphi_i-\dot{\varphi}_i^2\xi_i\sin\varphi_i+2\dot{\varphi}_i\dot{\xi}_i\cos\varphi_i\right) \tag{52}$$

where $i = 1,\dots,n$.

4. Numerical calculations

In this section we present the examples of synchronous configurations of the system (1) and (2) and the maps showing the dependence of the kind of the obtained synchronous configuration on the initial conditions. These results have been obtained by the numerical integration of Eqs. (5) and (6) (for the system composed of a beam and identical pendula) by the 4th order Runge–Kutta method.

The initial conditions are decisive as regards the kind of synchronization. The system under consideration has at least three degrees of freedom (depending on the number of pendula) so it has been decided to impose certain limitations on these conditions. It has been assumed that the numerical experiments start with the state in which the pendula suspended on the fixed beam move steadily performing oscillations described by function (14). For the initial time of $t = 0.0$, the state of the kth pendulum is described thus by the initial value of pendulum displacement φ_{i0} and the initial velocity ω_{i0}:

$$\begin{aligned}\varphi_{k0}&=\Phi\sin\beta_{k0}\\ \omega_{k0}&=\alpha\Phi\cos\beta_{k0}\end{aligned} \tag{53}$$

Such an assumption means that the initial state of the pendulum is described only by one value β_{k0}. As has been already mentioned, the initial conditions of the beam are as follows: y_0=0.0, v_0=0.0.

The dimensionless parameters of the sample systems are the following: the van der Pol's coefficients $c_{\varphi i}=-0.01$ and $c_{VDPi}=60.0$ (for these values, the amplitude of oscillations of the pendula suspended on the fixed beam is equal to $\Phi_i=0.25\approx15^\circ$), the beam mass $m_b=2.0$, the stiffness coefficient $k_y=10.0$, the damping coefficient of the beam motion c_y depends on the number of pendula and corresponds to the logarithmic decrement of the oscillations damping $\Delta=\ln1.2$ (the oscillator of the spring stiffness coefficient k_y and the mass equal to m_b+n – mass of the beam and the pendula).

4.1. Synchronization of two identical pendula

If two pendula of the same masses and the same periods of oscillations are connected to the beam, then there are possible three kinds of synchronization, shown in Fig. 2(a-c), as it follows from the above-presented formulas, namely:

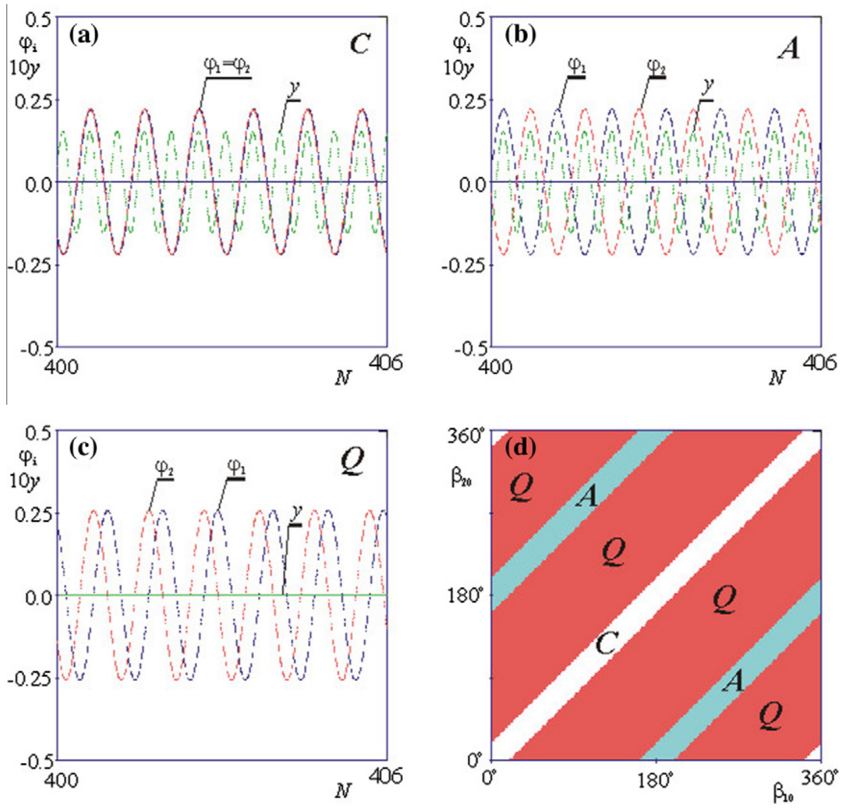

Fig. 2. Synchronization of the system with two pendula; (a) complete synchronization versus time, (b) antiphase synchronization versus time, (c) quarter-phase synchronization versus time, (d) basins of attraction of various kinds of synchronization.

- *complete synchronization* (Fig. 2(a)), during which both pendula move identically ($\varphi_1 = \varphi_2$) and the beam oscillates with a frequency twice as high as the frequency of the pendulum motion,
- *antiphase synchronization* (Fig. 2(b)), during which the motion of one pendulum is a mirror reflection of the motion of the second one ($\varphi_1 = -\varphi_2$) and the beam oscillates with a circular frequency twice as high as the circular frequency of the pendulum motion, identically as in the case of the complete synchronization,
- *quarter-phase synchronization* (Fig. 2(c)), during which the phase shift between the displacements of both pendula is equal to 90° and the fundamental (i.e., second) harmonics of the beam displacement is equal to zero, thus the beam is fixed practically.

The displacements of the pendula and the beam (magnified 10 times) in Fig. 2(a-c) are shown in the steady state as a function of time expressed by number N of the periods of free oscillations of the pendula suspended on the fixed beam. Fig. 2(d) shows the basins of attraction, i.e., the regions of the initial values β_{10} and β_{20}, which initiate individual synchronous configurations. As can be seen, the quarter-phase synchronization dominates. An occurrence of the complete or antiphase synchronization requires such initial values of the phase angles the difference of which is close to zero or 180°. Of course, the size of the basins of attractions of individual configurations depends on the beam parameters: m_b, k_y and c_y, which can be an object of further investigations.

4.2. Synchronization of three identical pendula

If three pendula of the same masses and the same periods of oscillations are connected to the beam, then four kinds of synchronization are possible, as shown in Fig. 3(a-d). It follows from the above-presented formulas, namely:

- *complete synchronization* (Fig. 3(a)), during which all the pendula move identically $\varphi_1 = \varphi_2 = \varphi_3$) and the beam oscillates with a frequency twice as high as the frequency of the pendulum motion,
- *antiphase synchronization* (Fig. 3(b)), during which the motion of one pendulum (for instance, φ_3) is a mirror reflection of the motion of the cluster composed of the remaining two pendula. The beam oscillates with a frequency twice as high as the frequency of the pendulum motion, identically as in the case of the complete synchronization,

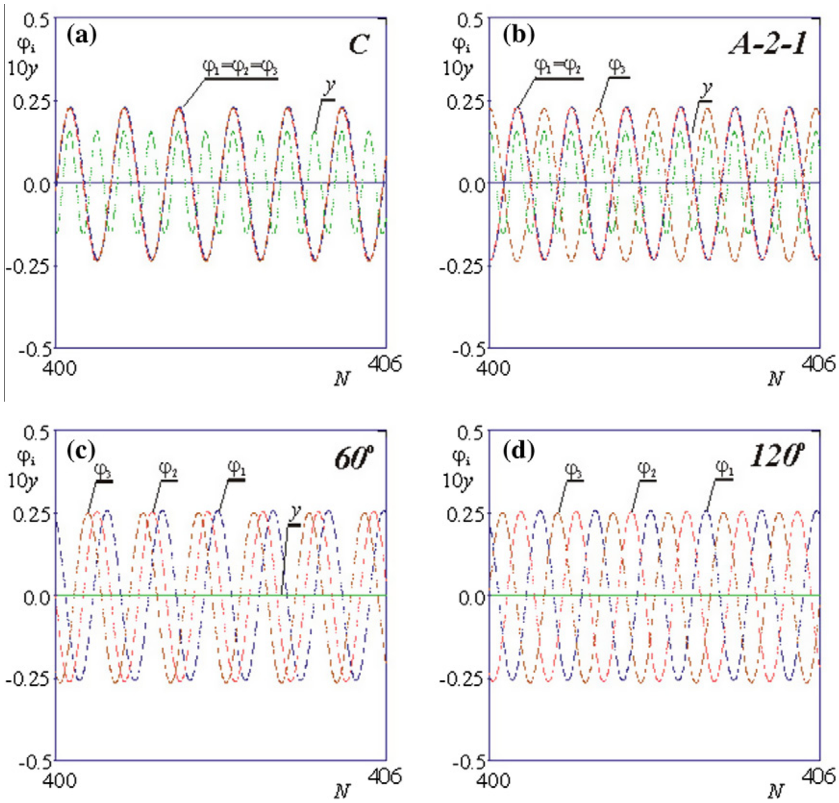

Fig. 3. Synchronization of the system with three pendula; (a) complete synchronization versus time, (b) antiphase synchronization versus time, (c) 60° – phase synchronization versus time, (d) 120° – phase synchronization versus time.

- *60° – phase synchronization* (Fig. 3(c)), during which the phase shifts between the displacements of the pendula are equal to 60° and the fundamental (i.e., second) harmonics of the beam displacement is equal to zero, thus the beam is fixed practically;
- *120° – phase synchronization* (Fig. 3(d)), during which the phase shifts between the displacements of the pendula are equal to 120° and the fundamental (i.e., second) harmonics of the beam displacement is equal to zero, thus the beam is fixed practically.

Fig. 4(a) presents an unstable configuration (*quarter-2-1 synchronization*), in which the motion of one pendulum (e.g., φ_1) is phase-shifted by 90° with respect to the displacement of the cluster composed of the remaining two pendula. This configuration can be observed when we assume the common initial value $\beta_{20} = \beta_{30}$; the solution to the disturbance equation

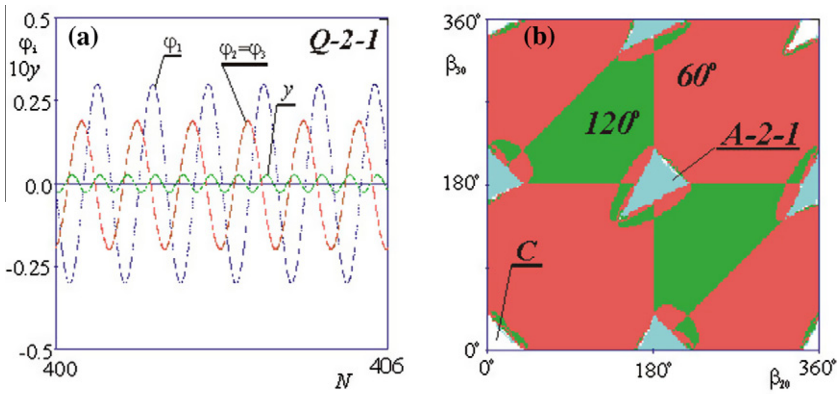

Fig. 4. Synchronization of the system with three pendula: (a) unstable quarter-phase synchronization versus time, (b) basins of attraction of various kinds of synchronization.

indicates that it is unsteady, however it is a steady synchronization in the system composed of two pendula, in which the mass of one pendulum is twice as high as the mass of the second one. Fig. 4(b) shows the regions of the initial values β_{20} and β_{30}, which initiate particular synchronous configurations. The initial value $\beta_{10} = 0°$. As can be seen, similarly as in the system with two pendula, we observe the domination of the 60° – and 120° – phase synchronizations, during which the beam remains practically fixed.

4.3. Synchronization of four identical pendula

If four pendula of the same masses and the same periods of oscillations are connected to the beam, then there are possible four kinds of synchronization, shown in Fig. 5(a-d), as it follows from the above-presented formulas, namely:

- *complete synchronization* (Fig. 5(a)), during which all the pendula move identically,
- two kinds of antiphase synchronization: *antiphase-3-1 synchronization* (Fig. 5(b)) and *antiphase-2-2 synchronization* (Fig. 5(c)), during which the motion of one cluster is a mirror reflection of the second cluster motion, clusters have one and three pendula, or two and two pendula, respectively. The beam oscillates with a circular frequency twice as high as the circular frequency of the pendulum motion, identically as in the case of the complete synchronization.
- *90°+90° – phase synchronization*) (Fig. 5(d)), during which we can observe two pairs of pendula in the state of the *quarter-phase synchronization*, shown in Fig. 2(c). A phase shift between these pairs is not explicitly defined: its value depends on the initial conditions. Notice that in a 1000 times multiplied displacement of the beam shown in Fig. 5(d); the fundamental (i.e., second) harmonics of the beam displacement is equal to zero, and, due to such a magnification, the next, fourth harmonics of the beam motion, which does not exist in the solution to the linearized equation of the beam motion (see Eqs. (18) and (19)), can be seen.

Fig. 6(a,b) shows the regions of the initial values β_{20} and β_{30}, which initiate individual synchronous configurations. The initial value $\beta_{10} = 0°$. The initial value $\beta_{40} = 45°$ in Fig. 6(a) and $\beta_{40} = 10°$ in Fig. 6(b). As can be seen, similarly as in the systems with two or three pendula, the 90°+90° phase synchronization during which the beam remains fixed in practice dominates.

Fig. 7(a,b) justifies the statement that the phase shift between the pairs of quarter-phase synchronized pendula does not have a fixed, constant value. In Fig. 7(a), we can observe the displacements of the pendula in the 90°+90°-phase

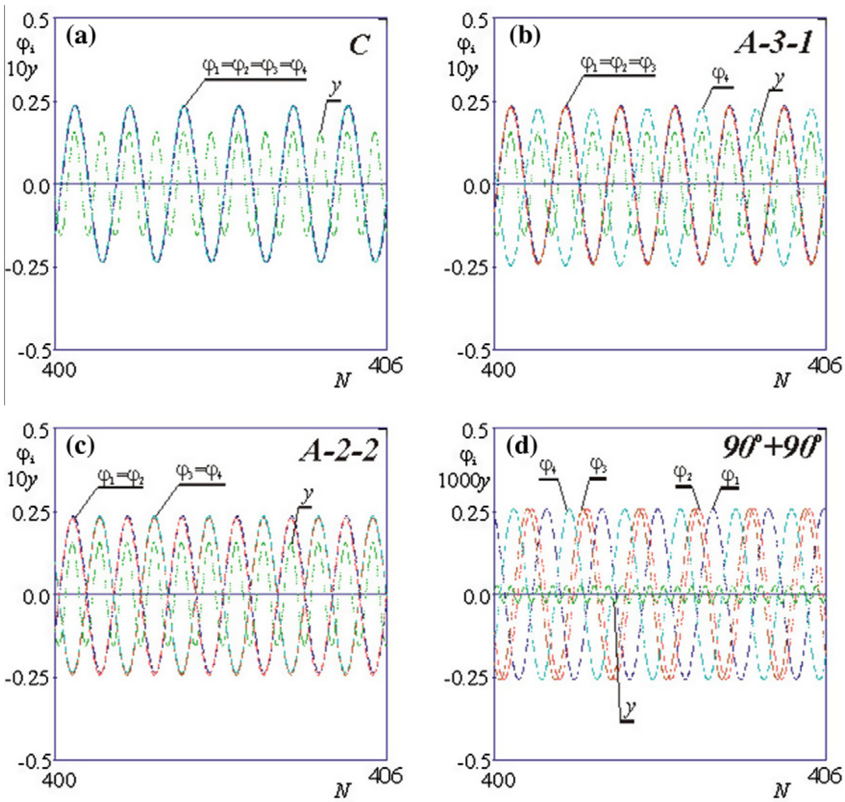

Fig. 5. Synchronization of the system with four pendula; (a) complete synchronization versus time, (b) antiphase 3-1 synchronization versus time, (c) antiphase 2-2 synchronization versus time, (d) 90°+90° – phase synchronization versus time.

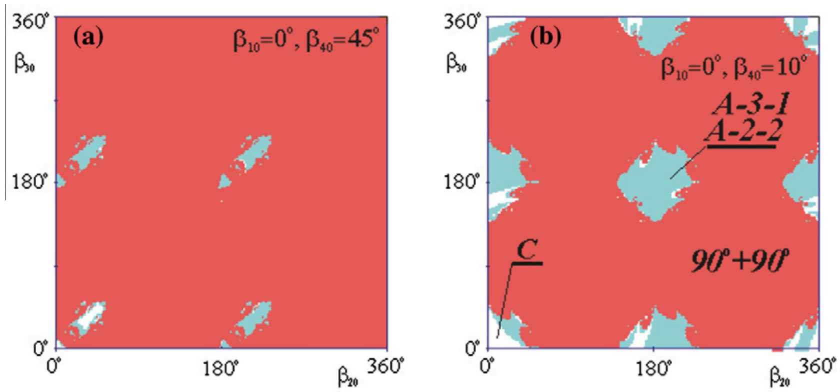

Fig. 6. Synchronization of the system with four pendula; (a) basins of attraction of various kinds of synchronization, $\beta_{10} = 0°$, $\beta_{40} = 45°$, (b) basins of attraction of various kinds of synchronization, $\beta_{10} = 0°$, $\beta_{40} = 10°$.

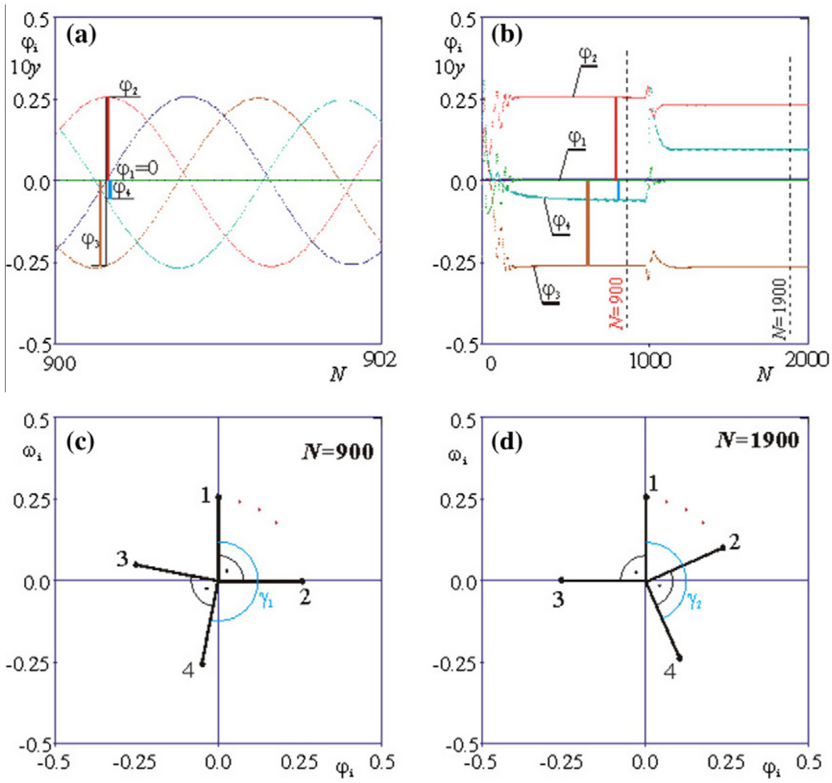

Fig. 7. Synchronization of the system with four pendula; (a) 90°+90° – phase synchronization versus time; the values of displacements presented in Fig. 7(b) are marked, (b) system motion with disturbances for *N* = 1000 versus time; positions at the instants $\varphi_1 = 0$, $\omega_1 > 0$, (c) Poincaré map showing the 90°+90° – phase synchronization, *N* = 900, (d) Poincaré map showing the 90°+90° – phase synchronization, *N* = 1900.

synchronization state, initiated by the following initial conditions: $\beta_{10} = 0°$, $\beta_{20} = 15°$, $\beta_{30} = 30°$, $\beta_{40} = 45°$. The displacements of the pendula at the instant when pendulum 1 goes with a positive velocity through the static equilibrium position are marked with vertical segments. Fig. 7(b) shows these displacements versus time, during 2000 periods of oscillations. A process of synchronization of the motion of the pendula, which for N = 900 assume the configuration shown in Fig. 7(c), can be seen. For *N* = 1000, a disturbance in the motion of the system occurred that consisted in the temporal switching off the van der Pol's damper of the first pendulum. When this damper is switched on again for *N* = 1030, the system motion is synchronized again in the configuration shown in Fig. 7(d). Fig. 7(c,d) shows the configurations of the system in the form of Poincaré maps on the phase plane; the state of pendula (ω_i, φ_i) can be seen at the instant when pendulum 1 goes with a positive velocity through the static equilibrium position ($\omega_1 = \Phi,\ \varphi_1 = 0$). In the light of the fact that the dimensionless circular frequency of pendulum oscillations is equal to 1, for the same ranges of axes of displacements (horizontal) and velocity (vertical), the graphical values of amplitudes of displacements and velocities are the same. As the result, the phase angles between the pendula can be read with a protractor from the maps, so the synchronization state (here: 90°+90° can be easily seen on them. In Fig. 7(c), pendulum 1 goes through the static equilibrium position, that is to say, its state is described by the point $(\omega_1 = \Phi,\ \varphi_1=0)$. Pendulum 2, which is synchronized with it, is shifted in phase by 90°, i.e., $(\omega_2 = 0,\ \varphi_2 = \Phi)$. Pendula 3 and 4 make the second pair of the synchronized pendula; the pairs are phase-shifted by tangle $\gamma_1 = 191^\circ$: $(\omega_4 = \Phi cos191^\circ,\ \varphi_4 = \Phi sin191^\circ)$, $(\omega_3 = \Phi cos281^\circ,\ \varphi_3 = \Phi sin281^\circ)$. The state of the system for N = 1900 is shown in Fig. 7(d). The state of pendulum 1 remains, of course, unaltered (it decides about the instant the state is recorded). The positions of the remaining pendula have changed. Pendulum 3 is now synchronized with pendulum 1. The second pair consisting of the synchronized pendula 2 and 4 is shifted with respect to the first pair by the angle $\gamma_2 = 155^\circ$: $(\omega_4 = \Phi cos155^\circ,\ \varphi_4 = 0\Phi sin155^\circ)$, $(\omega_2 = \Phi cos65^\circ,\ \varphi_4 = \Phi sin65^\circ)$. A further disturbance would result in the next alternation in the angle between the pairs of synchronized pendula. This result of the numerical experiment, which points out to the fact that four pendula always (despite the cases of the complete or antiphase synchronization) form two quarter-synchronized pairs, is convergent with the results of the solution to the system of Eq. (45): there is no other solution to this system than $\beta_{1\ldots4} = 0^\circ,\ 90^\circ,\ \gamma,\ \gamma + 90^\circ$, i.e., a solution of the type (here comes a faulty example, of course) than $\beta_{1\ldots4} = 0^\circ$, 95°, 150°, 246°.

4.4. Synchronization of five (or more) identical pendula

If five pendula are connected to the beam, then as can be easily expected, the following is possible:

- *complete synchronization*, during which all pendula move identically.
- two kinds of antiphase synchronization: *antiphase-3-2 synchronization* and *antiphase-4-1 synchronization*, during which the motion of one cluster is a mirror reflection of the motion of the second cluster; the clusters consist of three and two pendula, or four and one pendula, respectively. The beam oscillates with a frequency twice as high as the frequency of the pendulum motion, identically as in the case of complete synchronization.

In the case when the number of pendula is higher than four, condition given by Eq. (32) consists of a system of two equations with four or more unknowns and (as opposed to the above-described case of four pendula) they have an infinitely many solutions that are not combinations of the solutions known from the systems with two or three pendula. If, for instance, the

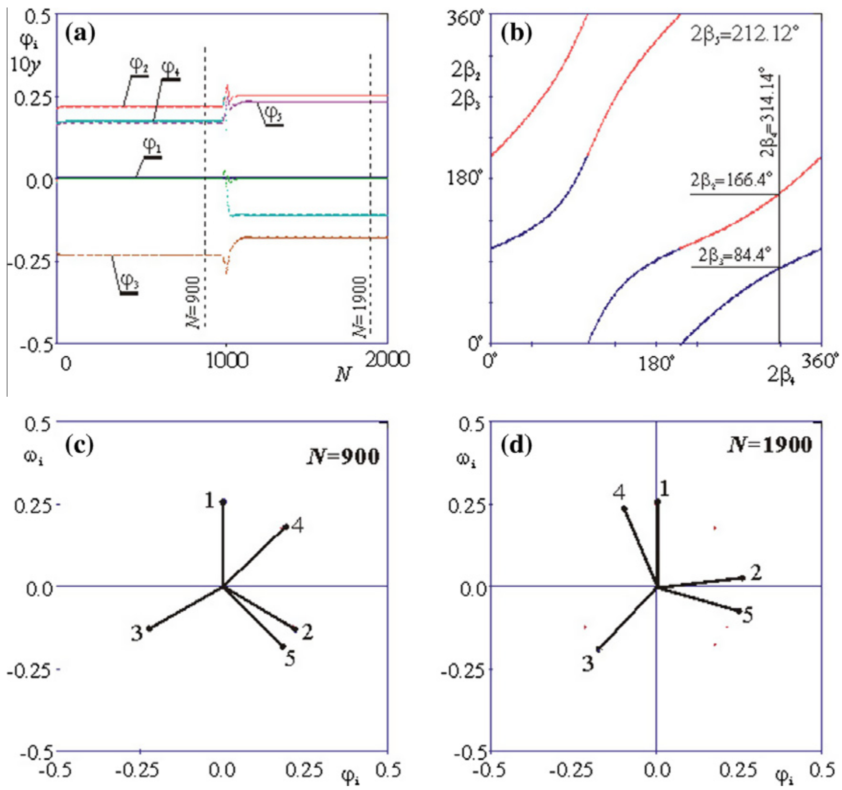

Fig. 8. Synchronization of the system with five pendula; (a) motion disturbance versus time for N = 1000, (b) solutions to system of Eq. (45), Poincaré map showing the configuration of the pendula before disturbance, for N = 900, (d) Poincaré map showing the configuration of the pendula after disturbance, for N = 1900.

initial conditions of the pendula are as follow: $\beta_{10} = 0^\circ,\ \beta_{20} = 120^\circ,\ \beta_{30} = 240^\circ,\ \beta_{40} = 45^\circ,\ \beta_{50} = 135^\circ (= 45^\circ + 90^\circ)$, then obviously we can observe a combination of the configuration of *120°-synchronization* of three pendula and a *quarter-phase-synchronization* of two pendula, but it is not a stable configuration: after a disturbance, we will no longer observe the combination of synchronous configurations of three and two pendula. The values of phase angles will fulfill the condition of Eq. (32), but the differences between them will no longer have the characteristic values of 120°, 60° or 90°.

Fig. 8(a-d) (analogous to Fig. 7(a-d)) represents an example of a reaction of the system with five pendula to a disturbance. In Fig. 8(a) (analogous to Fig. 7(b)), we can observe the displacements of the pendula recorded at the instances when pendulum 1 goes with a positive velocity through the static equilibrium position, initiated by the above-mentioned initial conditions ($\beta_{10} = 0^\circ$, $\beta_{20} = 120^\circ$, $\beta_{30} = 240^\circ$, $\beta_{40} = 45^\circ$, $\beta_{50} = 135^\circ$) as a function of time. A configuration of the pendula for $N = 900$ is shown in Fig. 8(c): we can check with a protractor that the phase angles between pendula 1, 2, 3 are equal to 120°,whereas the phase shift between pendula 4 and 5 equals 90°. For $N = 1000$, a disturbance in the system motion occurred that consisted in a switching off the van der Pol's damper of the first pendulum, which lasted for 30 oscillations. When this damper is switched on again for $N = 1030$, the system synchronizes again but in another configuration shown in Fig. 8(d). The following values of phase angles can be read from it: $\beta_1 = 0^\circ$, $\beta_2 = 83.22^\circ$, $\beta_{30} = 221.70^\circ$, $\beta_{40} = 337.07^\circ$, $\beta_{50} = 106.06^\circ$ (these values have been calculated on the basis of the text file of the numerical results – this is the reason for the accuracy up to one hundred part of the grade). These values do not correspond to any combination of synchronous configurations of two and three pendula. Fig. 8(b) shows the solutions to Eq. (32) for five pendula. It has been assumed that $2\beta_1 = 0^\circ$, $2\beta_{50} = 212.12^\circ$, whereas the angle $2\beta_4$ on the horizontal axis alters from zero to 360°. The blue and red lines represent the values of the angles $2\beta_2$ and $2\beta_3$, which together with the formerly defined $2\beta_1$, $2\beta_4$ and $2\beta_5$, fulfill condition (32). The figure shows that for $2\beta_4 = 674.14^\circ (= 314.14^\circ)$, the angles $2\beta_2 = 166.44^\circ$, $2\beta_3 = 443.40^\circ (= 84.40^\circ)$ read from Fig. 8(d) fulfill Eq. (32). The next disturbance would lead to a subsequent change in values of the phase angles. It means that in the system with five pendula (and a higher number of pendula as well) the phase synchronization, during which the beam is fixed, does not take place. There are no constant, characteristic values of phase angles between the group of two and the group of three pendula, regained by the system after the motion disturbance. The fact that we can observe a periodic motion again after the disturbance does not indicate synchronization as the periodicity of the motion results from the fact that the pendula are identical and not from the fact that there is a synchronizing mechanism (W^{SYN}) due to each the pendula exert an influence on one another.

To justify clearly the above-mentioned statement about the lack of the phase synchronization, Fig. 9 shows the behavior of the system with five pendula, which differs from the above-mentioned one only in this respect that the motion of the

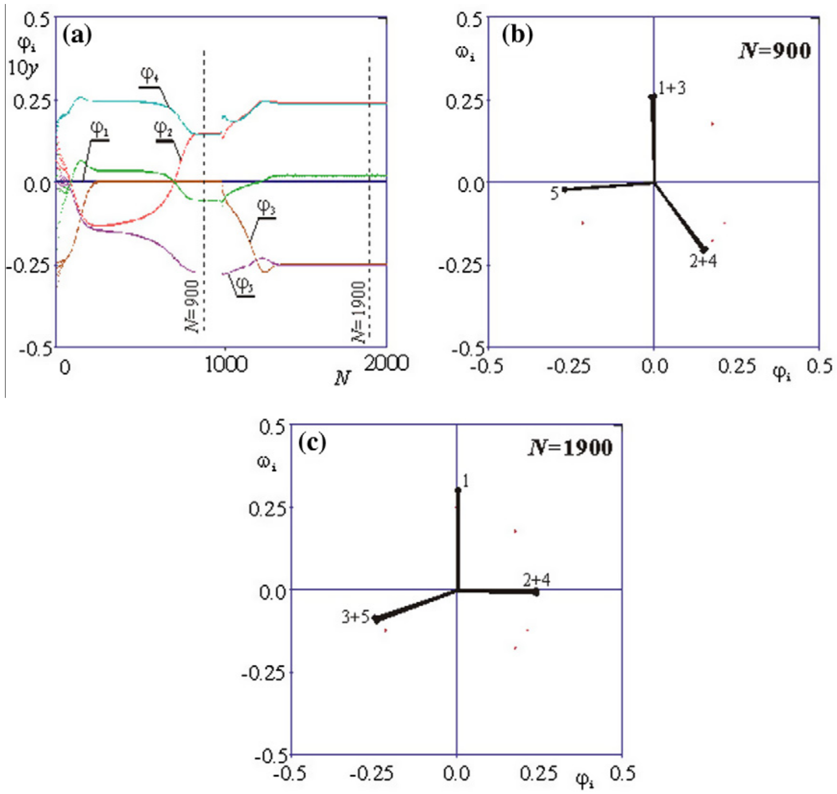

Fig. 9. Synchronization of the system with five pendula – the beam displaces horizontally; (a) motion disturbance versus time for $N = 1000$, (b) Poincaré map showing the synchronous configuration of the pendula before disturbance, for $N = 900$ – three clusters, (c) Poincaré map showing the synchronous configuration of the pendula after disturbance, for $N = 1900$ – three clusters again.

beam takes place along the horizontal direction and not the vertical one [4–6,10,14]. Beginning with the same initial conditions ($\beta_{10} = 0°$, $\beta_{20} = 120°$, $\beta_{30} = 240°$, $\beta_{40} = 45°$, $\beta_{50} = 135°$), the system attains the synchronous configuration shown in Fig. 9(b) – the pendula form three clusters. The first cluster consists of pendula 1+3, the second one – of pendula 2+4, and the third one is represented by a single pendulum 5. When the motion is disturbed for $N = 1000$, the pendula form a synchronous configuration composed of three clusters again. This time the first cluster is composed of one pendulum, the second one – of pendula 2+4, and the third one – of pendula 3+5. As can be seen, the disturbance is so strong that the clusters are decomposed, and the system returns to the configuration of three clusters composed of the same number of pendula and shifted by the same phase angles. Here, in the system with the beam displacing horizontally, we can see a phenomenon of the phase synchronization indeed. In the system with the beam displacing vertically – it is not the case.

5. Conclusions

If a series of identical van der Pol's pendula is suspended on the rigid beam that can displace along the vertical direction, then a phenomenon of synchronization of 2, 3 or 4 pendula can be observed that consists in establishing phase angles which have constant characteristic values. The resultant force the pendula exert on the beam is (in the linear approximation) harmonic, with a period twice as short as the period of pendulum oscillations or equal to zero (there is also its fourth harmonics, but only in solutions to nonlinear equations). Contrary to this case in the systems in which the beam displaced along the horizontal direction, this force had the first and third harmonics [4–6,10].

The following configurations of synchronized pendula have been identified, namely:

(i) the beam displaces with the "second harmonics",:
- complete synchronization when all the pendula displace identically, forming one cluster;
- antiphase synchronization of two clusters of a different, dependent on the initial conditions, number of pendula; independently of the number of pendula in these two clusters, the beam moves in the same way (its motion is identical as the motion which can be observed during the complete synchronization),

(ii) the forces the pendula act on the beam balance and the beam is (in the linear approximation) fixed:
- quarter-phase synchronization of two pendula shifted in phase by 90°;
- 60° – or 120° – synchronization of three pendula;
- 90°+90°-synchronization of four pendula: two pairs of pendula are in the state of the quarter-phase synchronization; the phase angle between these pairs is accidental: its value depends on the initial conditions; when the motion is disturbed, the system will return to the state of the 90°+90° – synchronization but this angle will change then.

The type of coupling considered in our studies is nonlocal [20] as each pendulum influences the motion of all other pendula through the motion of the beam. For the periodic motion of the pendula the observed phase shifted states can be considered as the special case of lag synchronization [15].

In the case when the system is composed of five or more pendula, a complete synchronization and antiphase synchronization of two clusters with a various number of pendula takes place. In such systems, the phase synchronization, during which the beam is at rest, has not been found; however, the suitable initial conditions lead to the state in which the forces the pendula exert on the beam balance out and the beam remains at rest, but the angles between the displacements of the pendula have accidental values, depending on the initial conditions and altering due to disturbances. The motion of the system is periodic of course, but this periodicity does not follow from the synchronizing role of the synchronization work W^{SYN}, but solely from the fact that identical pendula have the same periods of oscillations and the motion of the system composed of them is periodic even if the beam which is suspended on them is fixed and the pendula cannot affect the motion of one another – they cannot adapt mutually in phase. This conclusion is basically different from the conclusion based on the observation of the systems with many pendula suspended on the beam that displaces horizontally, where the phenomenon of combining the pendula in three (or seldom five) synchronized clusters occurs.

The results of the numerical simulations conducted for the nonlinear model of the mathematical system have confirmed the usefulness and correctness of the predictions based on the linearized equation of the beam motion, on the assumption of the harmonic motion of the pendula. The observed synchronous configurations are robust as they exist for the wide range of system parameters and initial conditions. They are stable in the presence of noise (unless the perturbation does not move the system out of the basin of attraction of the particular configuration).

Acknowledgments

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