

Chaos in coupled clocks

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We consider the dynamics of two pendulum clocks (with pendulums of the same length but different masses) suspended on the same beam. We give evidence that beside the complete and phase synchronizations the considered system can exhibit long period synchronization and chaotic behavior. We argue that the observed phenomena are robust.

Keywords: Huygens clocks, synchronization, chaos.

C. Huygens' observation of the antiphase synchronization of two pendulum clocks mounted together on the same beam [Huygens, C., 1665] was one of the first observations of the phenomenon of the coupled harmonic oscillators, which stimulated a great number of studies of the systems which can synchronize [Pikovsky, A., *et al.*, 2001; Blekham, I.I., 1988]. Recently, Huygens' original experiment has been rediscussed by a few groups of researchers [Pogromsky, A., *et al.*, 2003; Bennet, M., *et al.*, 2002; Senator, M., 2006; Dilao, R., 2009; Kumon, M., *et al.*, 2002; Fradkov, A.L. and Andrievsky, B., 2007, Pantaleone, J., 2002; Ulrichs, H., *et al.*, 2009; Czołczynski, K., *et al.* 2009a, 2009b]. To explain Huygens' observations special experiments have been performed [Bennet, M., *et al.*, 2002; Pantaleone, J., 2002; Czołczynski, K., *et al.* 2009a, 2009b]). It has been shown that to repeat Huygens' results, high precision (the precision that Huygens certainly could not achieve) is necessary.

In our previous paper [Czołczynski, K., *et al.* 2011] we consider the synchronization of two clocks which have pendulums with the same length (the same period of pendulums oscillations) but different masses. Such clocks are accurate, i.e., show the same time. We show that two such clocks hanging on the same beam

beside the complete (CS) and phase synchronizations (PS) already demonstrated in [Pogromsky, A., et al., 2003; Bennet, M., et al., 2002], perform the third type of synchronization in which both pendulums oscillate with the same period T^* . We identify period T^* to be significantly larger than the period of pendulums oscillations in the case when the beam is at rest $-T$. This type of generalized synchronization has been called a long period synchronization (LPS). Additionally, we show that beside the periodic synchronous behavior the clocks pendulums can perform chaotic-like uncorrelated oscillations.

In this paper we give evidence that two coupled clocks can show chaotic behavior. We study the sensitivity of (LPS) and chaotic behavior on the parameters of the escapement mechanism. We show that in the wide range of system parameters the system exhibits multistability, i.e., the coexistence of various (LPS) and chaotic attractors.

The synchronization of two clocks can be studied using the model shown in Figure 1. It consists of the rigid beam and two pendulum clocks suspended on it. The beam of mass M can move in a horizontal direction, its movement is described by coordinate x . The beam M is connected to the base by a linear spring k_x and linear damper c_x . The clocks' pendulum consists of the light beam of the length l and the mass mounted at its end. We consider the pendulums with the same length l but different masses m_1 and m_2 . The same length of both pendulums guarantees that the clocks are accurate, i.e., both show the same time. The motion of the pendulums is described by angles φ_1 and φ_2 and is damped by dampers (not shown in Figure 1) with damping coefficients c_{φ_1} and c_{φ_2} . The damping coefficients $c_{\varphi_{1,2}}$ are proportional to the pendulums' masses $m_{1,2}$. The pendulums are driven by the escapement mechanism described in details in [Czolczynski, K., *et al.*, 2009b; Rowlings, A.L., 1944; Lepschy, A. M., *et al.*, 1993; Roup, A.V., *et al.*, 2003]. Notice that when the swinging pendulums do not exceed certain angle γ_N , the escapement mechanisms generate constant moments M_{N1} and M_{N2} (proportional to the pendulum masses $m_{1,2}$).

This mechanism acts in two successive steps (the first step is followed by the second one and the second one by the first one). In the first step if $0 < \varphi_i < \gamma_N$ ($i=1,2$) then $M_{Di} = M_{Ni}$ and when $\varphi_i < 0$ then $M_{Di}=0$. For the second stage one has for $-\gamma_N < \varphi_i < 0$ $M_{Di}=-M_{Ni}$ and for $\varphi_i > 0$ $M_{Di}=0$. The energy supplied by the escapement mechanism balance the energy dissipated due to the damping. The parameters of this mechanics have been chosen in such the way that for the beam M at rest both pendulums perform oscillations with the same amplitude (the detailed description of the escapement mechanism has been given

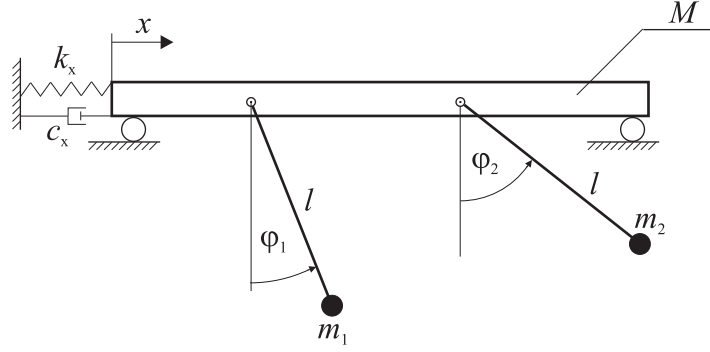


Fig. 1: The model of the system – two pendulum clocks are mounted to the beam which can move horizontally.

in our previous work [Czolczynski, K., *et al.*, 2009b]).

The equations of motion are as follow:

$$m_i l^2 \ddot{\varphi}_i + m_i \ddot{x} l \cos \varphi_i + c_{\varphi i} \dot{\varphi}_i + m_i g l \sin \varphi_i = M_{D_i}, \quad (1)$$

$$\left(M + \sum_{i=1}^2 m_i \right) \ddot{x} + c_x \dot{x} + k_x x + \sum_{i=1}^2 m_i l (\ddot{\varphi}_i \cos \varphi_i - \dot{\varphi}_i^2 \sin \varphi_i) = 0,$$

where $i = 1, 2$. Eqs (1) describes the dynamical system which performs the self-excited oscillations [Andronov, A., *et al.*, 1966].

In our numerical simulations eqs (1,2) have been integrated by the 4th-order Runge-Kutta method adopted for the discontinuous systems (the integration step has been decreased when the trajectory has been approaching discontinuity). The initial conditions have been set as follows; (i) for the beam $x(0) = \dot{x}(0) = 0$, (ii) for the pendulums the initial conditions $\varphi_1(0), \dot{\varphi}_1(0)$ have been calculated from the assumed initial phases β_{10} and β_{20} , i.e., $\varphi_1(0) = \Phi \sin \beta_{10}$, $\dot{\varphi}_1(0) = \alpha \Phi \cos \beta_{10}$, $\varphi_2(0) = \Phi \sin \beta_{20}$, $\dot{\varphi}_2(0) = \alpha \Phi \cos \beta_{20}$, where Φ and α ($\alpha = 2\pi/T$) are respectively the amplitude and the frequency of the pendulums when the beam M is at rest. We consider the following parameter values: $\gamma_N = 5.0^\circ$, $l = g/4\pi^2 = 0.2485$ [m], $M = 10.0$ [kg], $m_1 = 1$ [kg], $c_x = 1.53$ [Ns/m], $k_x = 3.94$ [N/m], $c_{\varphi 1} = 0.0083 \times m_1$ [Ns], $c_{\varphi 2} = 0.0083 \times m_2$ [Ns], $M_{N1} = 0.075 \times m_1$ [Nm], $M_{N2} = 0.075 \times m_2$ [Nm] and $m_2 \in [3.10, 4.27]$, for which system (1) exhibits the coexistence of (CS), (LPS) and chaotic behavior.

In Figure 2(a-d) we show Poincare maps for typical periodic and chaotic solutions. We plot position φ_2 and velocity $\dot{\varphi}_2$ of the second pendulum when the first pendulum has zero velocity $\dot{\varphi}_1 = 0$ and change its sign from positive to negative. An examples of (LPS) with periods $6T$ and $59T$ are shown Figure 4(a,b) and Figure 4(c) presents chaotic behavior. The pendulum trajectory of Figure 4(c) is characterized by the

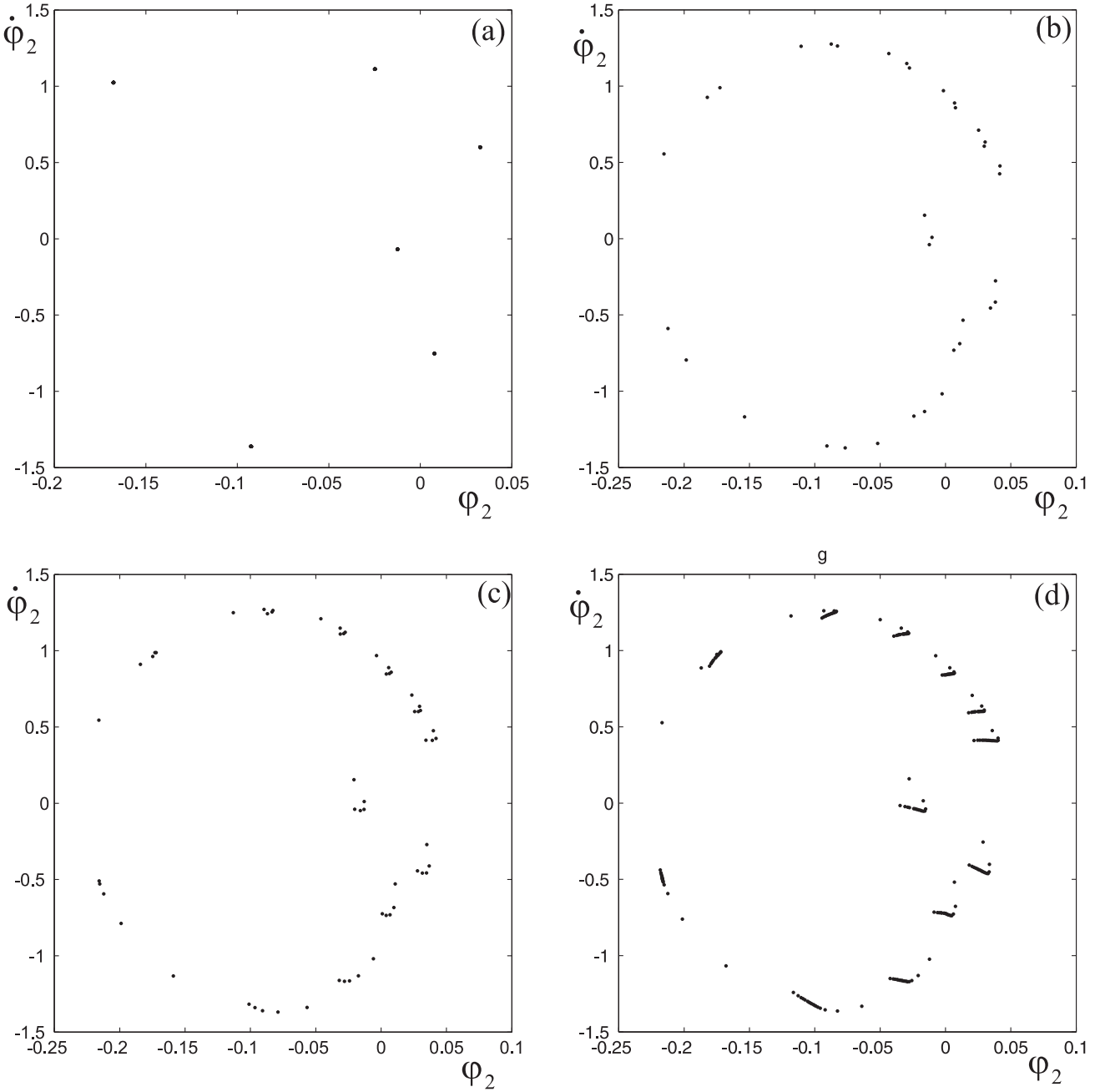


Fig. 2: Poincaré maps $(\varphi_2, \dot{\varphi}_2)$ for different (LPS) and chaotic attractors; $m_1 = 1.0$ [kg], $m_2 = 3.105$ [kg], $l = g/4\pi^2 = 0.2485$ [m], $M = 10.0$ [kg], $c_x = 1.53$ [Ns/m], $k_x = 3.94$ [N/m], $c_{\varphi_1} = 0.0083 \times m_1$ [Ns], $c_{\varphi_2} = 0.0083 \times m_2$ [Ns], $M_{N1} = 0.075 \times m_1$ [Nm], $M_{N2} = 0.075 \times m_2$ [Nm]: (a) $\gamma_N = 4.9^\circ$, $T = 6$, (b) $\gamma_N = 5.2^\circ$, $T = 35$, (c), $\gamma_N = 5.1^\circ$, $T = 59$, (d) $\gamma_N = 4.9^\circ$, chaotic behavior.

largest Lyapunov exponent equal to 0.127 (it has been estimated by the synchronization method described in [Stefanski, A. and T. Kapitaniak (2003), Kapitaniak, T. and Stefanski, A., 2003])¹. This calculations confirm that the coupled clocks can exhibit chaotic behavior ².

¹As eqs. (1) are discontinuous one cannot directly calculate the largest Lyapunov exponent from the linearized dynamics around the trajectory.

²Chaotic behavior of a single pendulum clock is also predicted and described in [Moon, F.C. and Stiefel, P.D., 2006] but model

Next for a given value of m_2 we consider the influence of the escapement mechanism parameters (γ_N and M_N) on the behavior of clocks' pendulums. We assume that the energy supplied to the system is the same in all cases, i.e., the product of $\gamma_N M_N$ is constant, so when we change γ_N we also recalculate M_N .

Notice that in the case of identical clocks (pendulums with the same masses) one can observe only two states: in-phase motion (complete synchronization (CS)) and anti-phase motion (phase synchronization (PS) with the phase shift equals to π). In Figure 2 we show a typical basin of attraction for identical clocks. Navy blue and yellow colors indicate (CS) and (PS) synchronizations respectively. When we take the close values of initial phases the system tends to the complete in-phase synchronization. This tendency is not significantly changed for nonidentical clocks (even with the large differences in pendulums' masses).

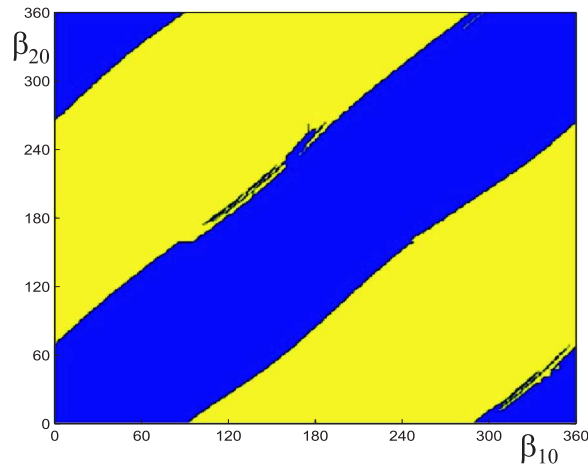


Fig. 3: Basins of attraction of synchronous states for $m_1 = m_2 = 1.0$, white color indicate complete (in-phase) synchronization, black color a anti-phase synchronization; $\Phi_1 \approx \gamma_N = 5.0^\circ$, $l = g/4\pi^2 = 0.2485$ [m], $M = 10.0$ [kg], $c_x = 1.53$ [Ns/m], $k_x = 3.94$ [N/m], $c_{\varphi 1} = 0.0083 \times m_1$ [Ns], $c_{\varphi 2} = 0.0083 \times m_2$ [Ns], $M_{N1} = 0.075 \times m_1$ [Nm], $M_{N2} = 0.075 \times m_2$ [Nm].

When the clocks are nonidentical they cannot exhibit antiphase synchronizations. In the initial conditions' domain in which identical clocks show antiphase synchronization (yellow in Figure 3) one observes coexistence of (PS), (LPS) synchronizations and chaotic motion of pendulums. Figure 4(a-f) shows the basins of attraction for $m_2 = 3.105$ and six different values of escapement mechanism parameter γ_N ($\gamma_N = 4.5^\circ$ (a); $\gamma_N = 4.8^\circ$ (b); $\gamma_N = 5.0^\circ$ (c); $\gamma_N = 5.05^\circ$ (d); $\gamma_N = 5.1^\circ$ (e); $\gamma_N = 5.2^\circ$ (f)). The numbers on the basins indicate the period of the (LPS) and (C) denotes the chaotic behavior. One can see the main range – period 1 (CS) synchronization remains the same in all six cases, so when the initial conditions

of the clock has been used.

of both pendulums are close to each other, then the parameters of the escapement mechanism have no influence on the pendulums' dynamics. For larger differences in initial conditions one can observe the (LPS) with different periods (in the range from 6T to 59T) and chaotic behavior. As it has been already mentioned all these phenomena are observed in the range of anti-phase synchronization observed for identical masses of both pendulums (see Figure 3). In Figure 4(a) one can observe a coexistence of 11T and 23T, 11T remains unchanged up to $\gamma = 5.2$, then in Figure. 4(b) 6T appears. In Figure 4(c) 6T attractor is dominant, then for $\gamma = 5.05$ 6T disappears and 12T (possible period doubling bifurcation of 6T), 30T and chaos (C) can be observed (Figure 4(d)). Then for $\gamma = 5.05$ (Figure 4(e)) all previous states are replaced by 59T LPS. In Figure 4(f) two new LPS ranges appear: 13T (replacing 11T) and 35T appear. The largest Lyapunov exponent of the chaotic attractors presented in Figure 4(a-f) varies between 0.095 and 0.125. Most of the (LPS) and (C) basins are small open sets of escapement mechanism parameters (practically small perturbation leads to their disappearance and the system jumps to another coexisting attractor).

To summarize we give evidence that two coupled clocks can show chaotic behavior, i.e., uncorrelated motion of the pendulums. We show that in the wide range of the system parameters the system exhibits multistability. The basins of attraction of the coexisting various (LPS) and chaotic attractors are small so practically any perturbation or fluctuation of the system parameters can result in the jumps of the system between different attractors. The described phenomena seem to be robust as they exist in the wide range of the system parameters.

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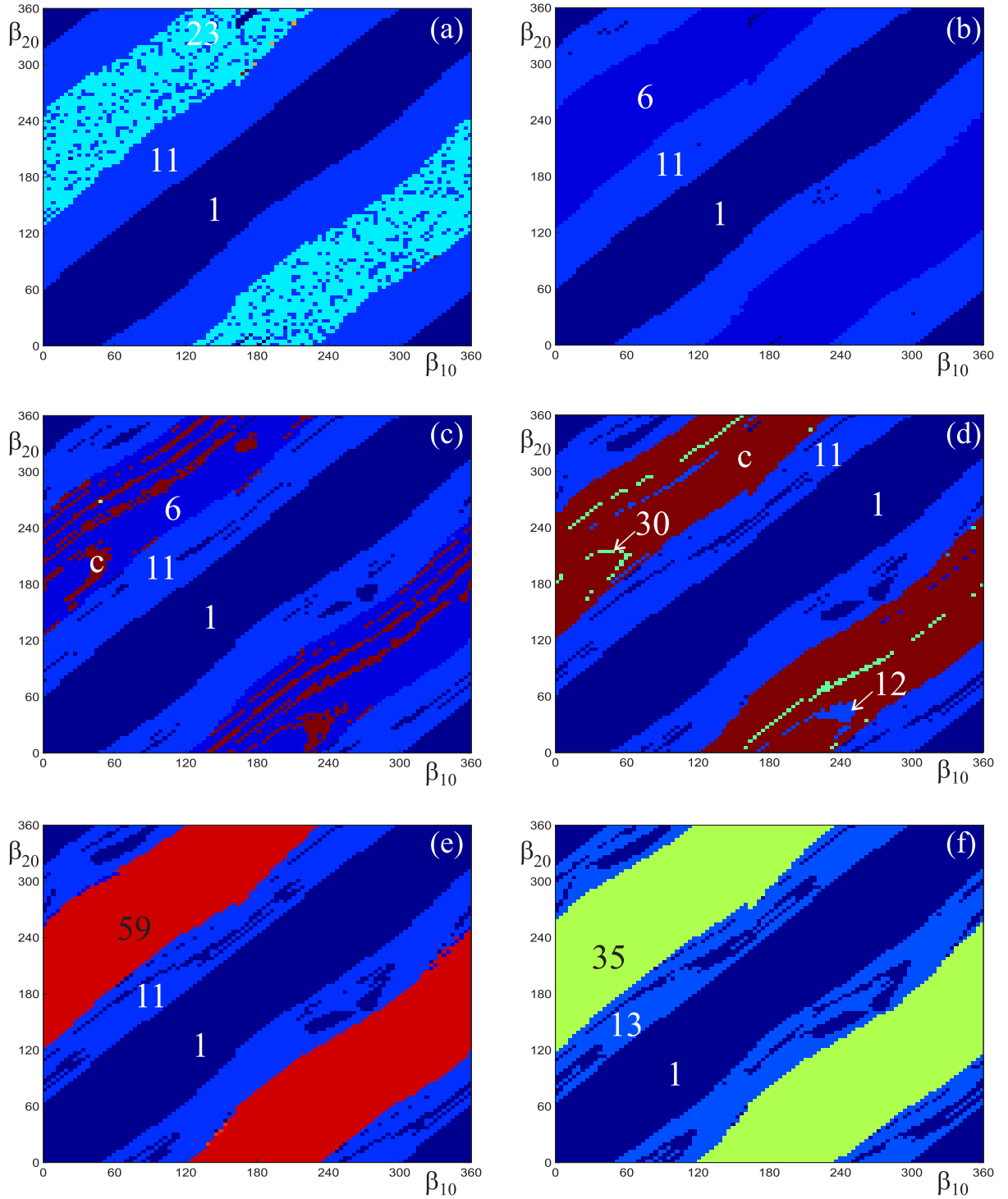


Fig. 4: Basins of attraction of different type of synchronous states for $m_2 = 3.105$; numbers indicate the period of the LPS, C shows chaotic behavior. Parameters: $\gamma_N = 4.5^\circ$ (a); $\gamma_N = 4.8^\circ$ (b); $\gamma_N = 5.0^\circ$ (c); $\gamma_N = 5.05^\circ$ (d); $\gamma_N = 5.1^\circ$ (e); $\gamma_N = 5.2^\circ$ (f). Other parameters are the same as in Fig. 2. Initial values: $x(0) = 0.0$, $\dot{x}(0) = 0.0$, $\varphi_{i0} = \Phi \sin \beta_{i0}$, $\dot{\varphi}_{i0} = \alpha \Phi \cos \beta_{i0}$.

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