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1:1 Mode locking and generalized synchronization in mechanical oscillators

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Abstract

We describe the relation between the complete, phase and generalized synchronization of the mechanical oscillators (response system) driven by the chaotic signal generated by the driven system. We identified the close dependence between the changes in the spectrum of Lyapunov exponents and a transition to different types of synchronization. The strict connection between the complete synchronization (imperfect complete synchronization) of response oscillators and their phase or generalized synchronization with the driving system (the (1:1) mode locking) is shown. We argue that the observed phenomena are generic in the parameter space and preserved in the presence of a small parameter mismatch.

1. Introduction

The phenomenon of synchronization in dynamical systems has been known for a long time [1]. The idea of synchronization has been also adopted for systems exhibiting chaotic behavior [2,3]. In recent years many different types of synchronization have been detected in nonlinear systems [2–7], starting from the simplest *complete synchronization* (CS), which takes place when the identical coupled systems exhibit identical, but still chaotic motion. This idea is quite well-known and has been described in many papers [7–10]. Pecora and Caroll have introduced a concept called the Master Stability Function (MSF) [11], which allows us to solve a problem of the CS for any set of connections, coupling weights and any number of oscillators. The concept of the CS has also been generalized for slightly different systems.

Recently, another type of synchronization, called the *phase synchronization* (PS) [6,7], has been detected in nonlinear systems. It can be described as synchronization of periodic oscillators, where only the phase locking is a necessity, whereas no requirements on amplitudes are imposed. The PS in nonlinear systems is defined as an appearance of the reaction between the phases of subsystems (or between the phase of subsystems and the driving signal), while the amplitudes can still be chaotic and uncorrelated.

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Rulkov et al. [12] presented an idea of the *generalized synchronization* (GS) of chaos. It is a type of cooperative behavior in unidirectionally coupled oscillators characterized by an existence of the stable and persistent functional dependence of response trajectories from the chaotic trajectory of the driving oscillator.

In mechanical systems the synchronization was discovered in the XVIIth century by the Dutch researcher, Christian Huijgens. He showed that a couple of mechanical clocks hanging from a common support were synchronized [13]. Some of the classical mechanical engineering applications are mentioned in Ref. [1]. In our previous works we study chaos synchronization in the coupled mechanical oscillators. In paper [9] we describe the complete synchronization between two coupled Duffing oscillators forced by the common harmonic signal. We show the typical behavior of the mechanical systems coupled by the spring and we also compared different methods of CS detection. In next paper [10] we developed the theory showing the ragged synchronizability (RSA), i.e., the existence of discontinuous regions of CS in system parameters' space. We show that in the mechanical systems where the coupling is realized by springs RSA is a common phenomenon. Additionally, we prove that in the small world networks [14,15], RSA could lead to decay of synchronizability. The relation between the maximum Lyapunov exponent of the coupled system and its coupling coefficient is determined in Ref. [16]. We presented the algorithms of the CS detection in the systems forced by the chaotic or discontinuous signal.

Generally, the above-mentioned papers concentrate on the CS phenomenon. The idea we propose in this paper is a new global view on the coupled dynamical systems driven by the chaotic signal. We focus our attention on a relation between the complete, generalized, phase synchronization and Lyapunov exponents. We prove their close relation. We also show that for the systems with mismatches, *the imperfect complete synchronization* (ICS) occurs when the oscillators are in the PS with the chaotic excitation.

The paper is organized as follows. In Section 2 we recall some theoretical basis. Section 3 describes the model the we have used in numerical investigations and the spectrum of its Lyapunov exponents. In Section 4 we present a few numerical examples of synchronization in identical or slightly different systems. Finally, we summarize our results in Section 5.

2. Complete, generalized and phase synchronization

2.1. Complete synchronization

The CS can appear only in the case of identical coupled systems, i.e., defined by the same set of ODEs with the same values of system parameters, say $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$. The CS takes place when all trajectories representing the coupled systems in the phase space, converge to the same value and remain in step with each other during further evolution. Hence, for two arbitrarily chosen trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$, we have

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0, \tag{1}$$

whereas in the case of slightly different coupled systems (the same ODEs with small mismatches of parameters), the ICS can be observed, i.e.:

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| \leqslant \varepsilon, \tag{2}$$

where ε is a vector of the small parameter.

2.2. Generalized synchronization

The GS was proposed by Rulkov et al. [12] as a generalization of the synchronization idea for unidirectionally coupled systems:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{3a}$$

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{h}(\mathbf{x})), \tag{3b}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{h}(\mathbf{x})$ is a function characterizing the coupling between the drive (3a) and the response (3b) system. We can say that the GS of these systems occurs if there exists a static functional relation

ψ between their states, i.e., $\mathbf{y}(t) = \psi[\mathbf{x}(t)]$. In order to detect the presence of the GS, a numerical method called the *mutual false nearest neighbors* has been proposed, which is based on the auxiliary system approach [17]. According to this method, the criterion for the GS existence is an appearance of the CS between the response subsystem (3a) and its identical replica, i.e., $\lim_{t\to\infty}\|\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{01}) - \mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{02})\| = 0$ where $(\mathbf{x}_0,\mathbf{y}_{01})$ and $(\mathbf{x}_0,\mathbf{y}_{02})$ are two generic initial conditions of system (3a) and (3b). The stability of the synchronization manifold of both response subsystems can be determined by means of the Lyapunov function [18] or Lyapunov exponents [2,19]. The GS problems have been researched both in the context of identical (when separated) systems (3a) and (3b) [2], and also in the cases when the response system is slightly (the same set of ODEs with different values of system parameters) or strictly different (another set of ODEs) than the driving oscillator [12].

2.3. Phase synchronization

In order to explain this idea, let us consider two autonomous systems defined in the 3D phase space $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, p)$ and $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, p+\delta)$, where $\mathbf{x}, \mathbf{y} = (x_{1,2}, y_{1,2}, z_{1,2})^T$, $p \in \mathbb{R}$ is a system parameter and δ describes the parameter mismatch. One can detect a simple way to define the phase of each system as $\phi_{1,2} = \arctan(u_{1,2}/z_{1,2})$, where $u_{1,2} = \sqrt{x_{1,2}^2 + y_{1,2}^2}$. The considered systems are in the PS if $\phi_1 = \phi_2$.

The occurrence of the PS in the autonomous systems is connected with the spectrum of Lyapunov exponents. Such cases have been studied in papers [6,7]. For example, consider an attractor of the single Rössler system working in the chaotic range which is characterized by one positive, one negative, and one zero Lyapunov exponent. When two Rössler systems with slight mismatches in parameters are coupled, with the increase of the coupling coefficient, the positive and negative exponents remain, whereas one of the zero exponents becomes negative and the PS occurs.

In our numerical investigations, to detect phase locking areas, we apply a method based on the mean frequency definition [2]:

$$\omega = \lim_{t\to\infty} 2\pi \frac{N_t}{t}, \tag{4}$$

where N_t is the number of the crossing Poincare section during the observation time t. The classical notation of the mode locking for periodic oscillators with a phase $\phi_{1,2}$ is

$$|n\phi_1 - m\phi_2| < \text{const}, \tag{5}$$

where n and m are integer numbers. The frequencies ($\omega_{1,2} = \dot{\phi}_{1,2}$) of the coupled oscillators are also locked, i.e., $n\omega_1 - m\omega_2 = 0$, and, in general, $n:m$ mode locking can occur. In this paper, we restrict ourselves to the case $n = m = 1$ due to the chaotic driving.

3. Model of the system and its Lyapunov exponents

General scheme of the system under consideration is shown in Fig. 1. This is a typical example of unidirectional coupling, where n-dimensional master $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ system drives two identical m-dimensional response oscillators $\dot{\mathbf{y}}_1 = \mathbf{g}(\mathbf{y}_1)$ and $\dot{\mathbf{y}}_2 = \mathbf{g}(\mathbf{y}_2)$ when separated. Its mathematical description is as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{6a}$$

$$\dot{\mathbf{y}}_1 = \mathbf{g}(\mathbf{y}_1) + \mathbf{q}\mathbf{h}(\mathbf{x}) + \mathbf{D}(\mathbf{y}_2 - \mathbf{y}_1), \tag{6b}$$

$$\dot{\mathbf{y}}_2 = \mathbf{g}(\mathbf{y}_2) + \mathbf{q}\mathbf{h}(\mathbf{x}) + \mathbf{D}(\mathbf{y}_1 - \mathbf{y}_2), \tag{6c}$$

where $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{y}_{1,2} \in \mathbf{R}^m$ represent the drive and response systems, respectively, $\mathbf{D}$ is $m \times m$ coupling matrix and $\mathbf{h}(\mathbf{x})$ is the same driving function as in Eq. (3b). Thus, the dynamics of the entire system is characterized by the spectrum of $(n+2m)$ Lyapunov exponents. Such a spectrum can be divided into two sets [16]:

(1) a set of n-number of *driving Lyapunov exponents* (DLEs) connected with the driving system,
(2) a set of $2m$-number of *response Lyapunov exponents* (RLEs) characterizing the dynamics of the response oscillators.

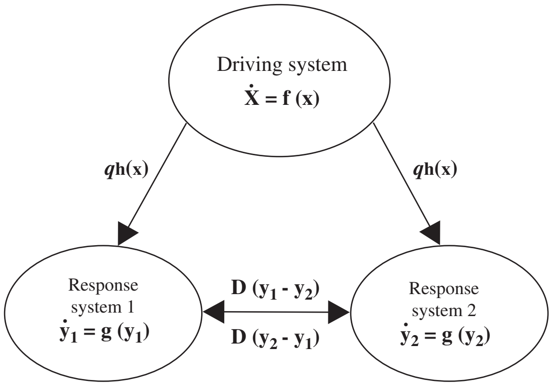

Fig. 1. General scheme of the system.

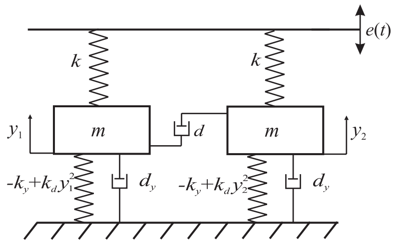

Fig. 2. Two identical Duffing oscillators excited by a common signal.

Among the RLEs we can again distinguish two subsets of the equal number of elements: *conditional Lyapunov exponents* (CLEs—λ_i^C, $i = 1, 2, \ldots, m$) [3], quantifying the GS between the drive and the response oscillators, and *transversal Lyapunov exponents* (TLEs—λ_i^T, $i = 1, 2, \ldots, m$) [20] determining the CS between the response systems. The negativity of all the CLEs or TLEs is a necessary condition for the occurrence of the GS or CS, respectively. In the case of the lack of diffusive coupling between response oscillators, i.e. coupling matrix $\mathbf{D} = \mathbf{0}$ in Eqs. (6b) and (6c), both of these LEs subsets are identical ($\lambda_i^C = \lambda_i^T$). Then, according to *auxiliary system approach* [3], the GS with the drive and CS of responses appear together.

As a real equivalent of the system (Eqs. (6a–c)) we consider two Duffing oscillators excited by the signal $e(t)$ shown in Fig. 2. Its evolution is described by

$$\begin{aligned} m\ddot{y}_1 + d_y\dot{y}_1 + (-k_y + k_d y_1^2)y_1 + d(\dot{y}_1 - \dot{y}_2) - k(e(t) - y_1) = 0, \\ m\ddot{y}_2 + d_y\dot{y}_2 + (-k_y + k_d y_2^2)y_2 + d(\dot{y}_2 - \dot{y}_1) - k(e(t) - y_2) = 0, \end{aligned} \tag{7}$$

where m, d_y, k_y, k_d, are mass, viscous damping, linear and nonlinear parts of the stiffness of the spring, respectively; d is a viscous damping of the dissipative coupling and $e(t)$ is a signal of excitation which is transmitted by the spring with a stiffness k. The derivatives in Eq. (8) are calculated with respect to non-dimensional time τ.

Having involved a dimensionless form of Eq. (7), we obtain the equation:

$$\begin{aligned} \ddot{\xi}_1 + c_1\dot{\xi}_1 + (-1 + c_2\xi_1^2)\xi_1 + d_1(\dot{\xi}_1 - \dot{\xi}_2) - k_1(c_3(\tau) - \xi_1) = 0, \\ \ddot{\xi}_2 + c_1\dot{\xi}_2 + (-1 + c_2\xi_2^2)\xi_2 + d_1(\dot{\xi}_2 - \dot{\xi}_1) - k_1(c_3(\tau) - \xi_2) = 0, \end{aligned} \tag{8}$$

where $\omega = k_y/m$, $c_1 = d_y/m\omega$, $c_2 = k_d m y_{st}^2/\omega^2 k_y^2$, $c_3(\tau) = e(t)/y_{st}$, $d_1 = d/m\omega$, $k_1 = k/m\omega^2$, $y_{st} = mg/k_y$ and $\tau = \omega t$. In our numerical analysis, we have assumed $c_2 = 1.0$, $k_1 = 0.1$, c_1 and d_1 are control parameters. As an external excitation, we have chosen a signal from the Rössler oscillator, given by the dimensionless equation:

$$\begin{aligned}\dot{x}_1 &= -\omega_R x_2 - x_3,\\ \dot{x}_2 &= \omega_R x_1 + a_R x_2,\\ \dot{x}_3 &= f_R + x_3(x_1 - c_R).\end{aligned} \tag{9}$$

The system under consideration (Eqs. (8)) is excited using the signal given by the variable x_1 from Eqs. (9), i.e. $c_3(\tau) = x_1$. We have introduced the parameters of the Rössler oscillator according to Rosenblum et al. [6]: $\omega_R = 1.0$, $a_R = 0.15$, $f_R = 0.2$ and $c_R = 10.0$, thus the evolution of chaotic excitation in the 3D driving subspace is characterized by the spectrum of three DLEs ($\lambda_1^D = 0.088$, $\lambda_2^D = 0.000, \lambda_3^D = -9.783$), which are insensitive to the variation of parameters of the response oscillators due to unidirectional coupling.

4. Numerical examples

4.1. Identical systems

In this section we present numerical simulations of the system demonstrated above. We start from showing a plot of Lyapunov exponents in the control parameter subspace (Fig. 3). The system given by Eqs. (8), forced by the Rössler oscillator (Eqs. (9)) is characterized by a set of seven Lyapunov exponents, but as we have mentioned before, three of them are DLEs connected with the Rössler oscillator. The remaining four RLEs describe the evolution of system (8) in the four-dimensional response subspace (two CLEs—λ_1^C, λ_2^C and two TLEs—λ_1^T, λ_2^T). In the considered case two of the RLEs characterizing Duffing subsystems are always negative ($\lambda_2^C < 0$, $\lambda_2^T < 0$), so these exponents have no influence on the synchronization and there is no necessity to present them. The remaining largest RLEs (λ_1^C and λ_1^T) shown in Fig. 3 have the crucial significance for the occurrence of synchronization. Their sign changes from positive to negative values. Our calculations have been performed for the following initial conditions: $\xi_1 = 0.1$, $\xi_2 = 0.11$, $\dot{\xi}_1 = 0.0$, $\dot{\xi}_2 = 0.0$, $x_1 = 0.1$, $x_2 = 0.0$, $x_3 = 0.0$. As in the considered system we have not observed coexisting attractors the calculations shown in Fig. 3 are not sensitive to the initial conditions.

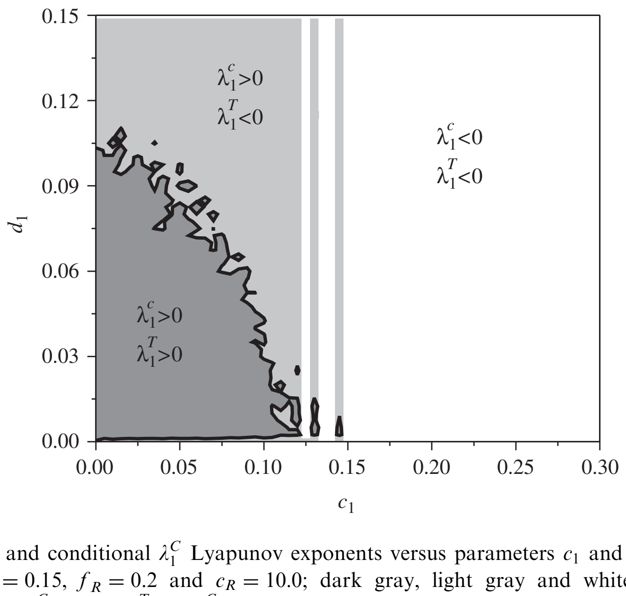

Fig. 3. The sign of transverse λ_1^T and conditional λ_1^C Lyapunov exponents versus parameters c_1 and d_1 for Eq. (8) forced by Eq. (9): $c_2 = 1.0$, $k_1 = 0.1$, $\omega_R = 1.0$, $a_R = 0.15$, $f_R = 0.2$ and $c_R = 10.0$; dark gray, light gray and white domains indicate, respectively, combinations: $\lambda_1^T > 0$, $\lambda_1^C > 0$, $\lambda_1^T < 0$, $\lambda_1^C > 0$ and $\lambda_1^T < 0$, $\lambda_1^C < 0$. Our calculations have been performed for the following initial conditions: $\xi_1 = 0.1$, $\xi_2 = 0.11$, $\dot{\xi}_1 = 0.0$, $\dot{\xi}_2 = 0.0$, $x_1 = 0.1$, $x_2 = 0.0$, $x_3 = 0.0$.

The dark gray area corresponds to the positive RLEs, where the completely desynchronous regime dominates, i.e., none of the synchronization types (CS, PS, GS) appear. Aside from this area, one ($\lambda_1^C < 0$ in the light gray region) or both (white area) RLEs are lower than zero and there the Duffing oscillators reach the CS (and obviously the PS) with each other. However, the common dynamics of the subsystems does not mean a correlation with the master system in the entire space of the parameters (d_1, c_1).

In order to study such a case in more detail, let us analyze the next two figures (Figs. 4 and 5). They show the cross-sections of the plane (d_1, c_1)—(Fig. 3) for arbitrarily chosen values of d_1 where we have proven a

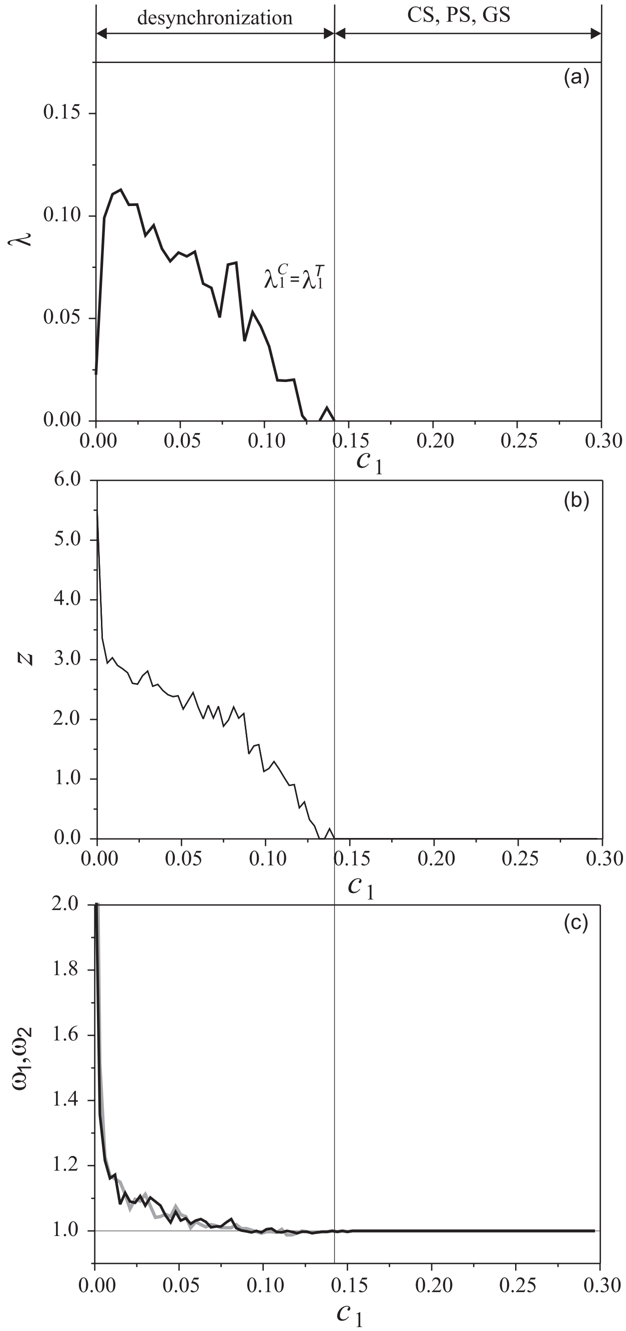

Fig. 4. Plots of the largest RLEs (a), synchronization error z (b) and phase ω_1 and ω_2 (c) versus c_1 for Eq. (8) forced by Eq. (9): $d_1 = 0.0$, $c_2 = 1.0$, $k_1 = 0.1$, $\omega_R = 1.0$, $a_R = 0.15$, $f_R = 0.2$ and $c_R = 10.0$. Regions of desynchronization and CS, PS or GS are indicated.

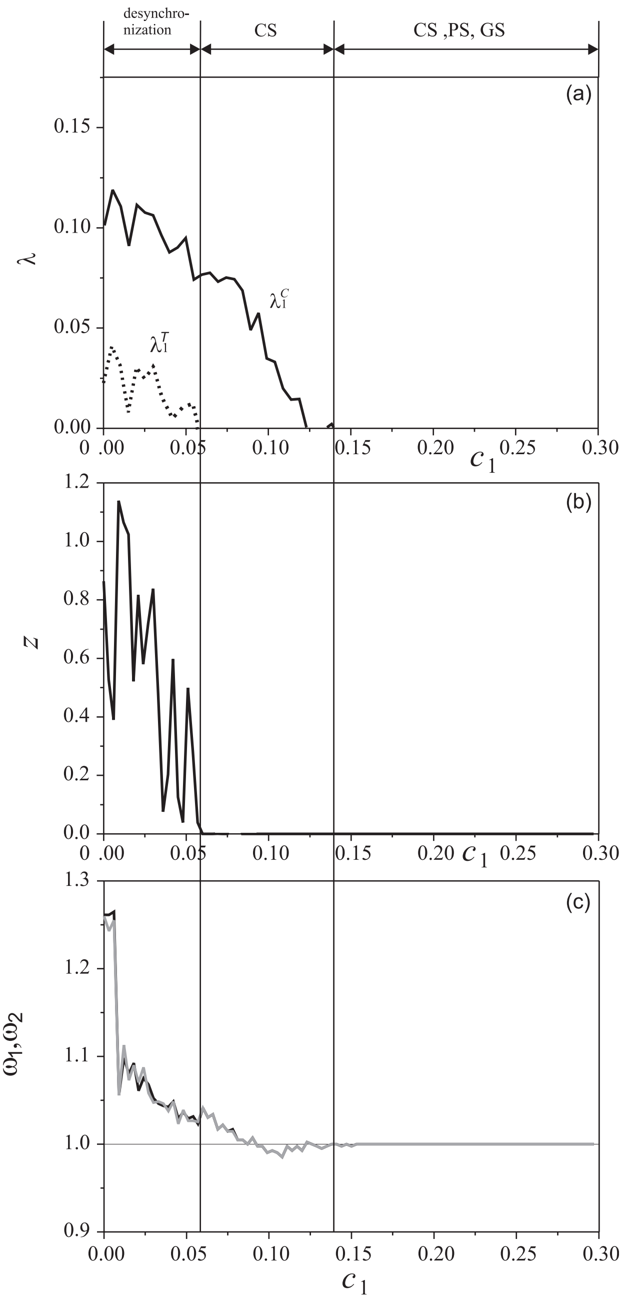

Fig. 5. Plots of the largest RLEs (a), synchronization error z (b) and phase ω_1 and ω_2 (c) versus c_1 for Eq. (8) forced by Eq. (9): $d_1 = 0.08$, $c_2 = 1.0$, $k_1 = 0.1$, $\omega_R = 1.0$, $a_R = 0.15$, $f_R = 0.2$ and $c_R = 10.0$. Regions of desynchronization and CS, PS or GS are indicated.

dependence of λ_1^C, λ_1^T, the synchronization error and the PS, respectively. The synchronization error fulfils the equation

$$z = \sqrt{(\xi_1 - \xi_2)^2 + (\dot{\xi}_1 - \dot{\xi}_2)^2}. \tag{10}$$

Values of phases ω_1 and ω_2 allow the detection of the PS. The phase of Rössler ϕ is normalized to 1 and the phases of Duffing oscillators ω_1 and ω_2 are calculated using Eq. (4) and normalized in the same manner as ϕ. When $\phi = \omega_1 = \omega_2 = 1$ one can observe the (1:1) mode locking.

The first cross-section for $d_1 = 0$ is shown in Fig. 4. This is a case of no diffusive coupling between the subsystems. Hence, we have $\lambda_1^C = \lambda_1^T$ and simultaneously a transition to the CS, PS and GS occurs at the same value of $c_1 \cong 0.14$. The CS appears, i.e., the synchronization error approaches zero (Fig. 4b), when $\lambda_1^T < 0$ (Fig. 4a). Also the GS between the drive (Rössler) and response (Duffing) systems occurs when $\lambda_1^C < 0$ (Fig. 4a). Moreover, the PS between them takes place if $\omega_{1,2} = 1$ (1:1 mode locking—see Fig. 4c).

The second cross-section presented in Fig. 5 has been made for $d_1 = 0.08$. This value of d_1 is in the middle of the range (Fig. 3), where the coupling of subsystems has a significant influence. In Fig. 5a two RLEs are shown. The first λ_1^T is corresponding to the CS of the Duffing oscillators (Fig. 5b) and λ_1^C to the PS and GS with Rössler (Fig. 5c). As can be seen, when $\lambda_2^C > 0$, the phases of the Duffing oscillators are uncorrelated with driving ($\omega_{1,2} \neq 1$) although they have common dynamics between $0.055 < c_1 < 0.14$. Thus, in the light gray area in Fig. 3, the GS and PS do not occur in spite of the observed CS between the response oscillators. A complete coincidence of the considered synchronous regimes exists only in the white region of the control parameter space (Fig. 3).

4.2. Nonidentical systems

In this subsection we analyze more real case of the considered system. It is obvious that in such systems it is impossible to achieve identity. Even the best made parts have some tolerance of form, attitude, etc. So, due to this property, we have decided to examine the system under consideration in this respect. In mechanical systems, the springs are the objects which cause a lot of problems, it is very hard to manufacture identical parts, so we assume a parameter mismatch in stiffness of the springs. We have chosen one value of a mismatch $\Delta = 1\%$ in the parameter c_2.

In this subsection we compare what happens to the mode locking and the synchronization error when such a small mismatch is applied. In Fig. 8 we show a cross-section corresponding to Fig. 4—$d_1 = 0.0$. In Fig. 6a we present a plot of four RLEs, then in Fig. 6b we show the phases of the system with mismatches and, at last, Fig. 6c presents the synchronization error z. For non-identical oscillators, the stable synchronized manifold $\xi_1 = \xi_2$ does not exist, so the TLEs cannot be defined in this case. Hence, all the RLEs are now the CLEs. As can be seen, slightly different subsystems reach the ICS after they synchronize their phases with excitation (Rössler system) and the GS between them appears (negative CLEs—see Fig. 6a). So, according to this behaviour, we can connect the ICS among uncoupled subsystems with the PS and GS with excitation.

Fig. 7 shows the cross-sections corresponding to Fig. 5, where a diffusive coupling between subsystems is added (the sequence of plots is the same as in Fig. 6). The synchronization error is smaller than in Fig. 6, but it is noteworthy that the inner coupling between the Duffing oscillators has no influence on their ICS, contrary to the identical systems (see Figs. 4 and 5). Only the PS with excitation decides about the ICS of subsystems. Comparing Figs. 6b, 7b with the corresponding Figs. 6c, 7c, some discrepancy between the PS (also GS) and the ICS can be observed. Namely, in the range $c_1 \in (\sim 0.14, \sim 0.18)$ the ICS does not take place (the synchronization error is relatively large) in spite of the presence of the PS here. The most likely explanation is an occurrence of the *lag phase synchronization* (LPS) in this range [7].

For c_1 larger than 0.18 where ICS take place ε is smaller than 0.022 (see Eq. (11)) and practically not visible in Figs. 6b and 7b.

4.3. Analysis of mismatches in parameter

As we mention before the CS is impossible in the case of non-identical coupled systems. However the synchronization error remains relatively small when the ICS occurs. The crucial parameter for quantifying the ICS is ε. The value of ε should be related to the maximum synchronization error sup(z) (Eq. (10)) and the parameter mismatch Δ. Our studies show that this relation can be approximated as follows:

$$\varepsilon \approx \frac{\sup(z) * \Delta}{N}, \qquad (11)$$

where N is equal to number of considered oscillators in network ($N = 2$). Relation (11) should be treated as an approximate calculation of the boundary value.

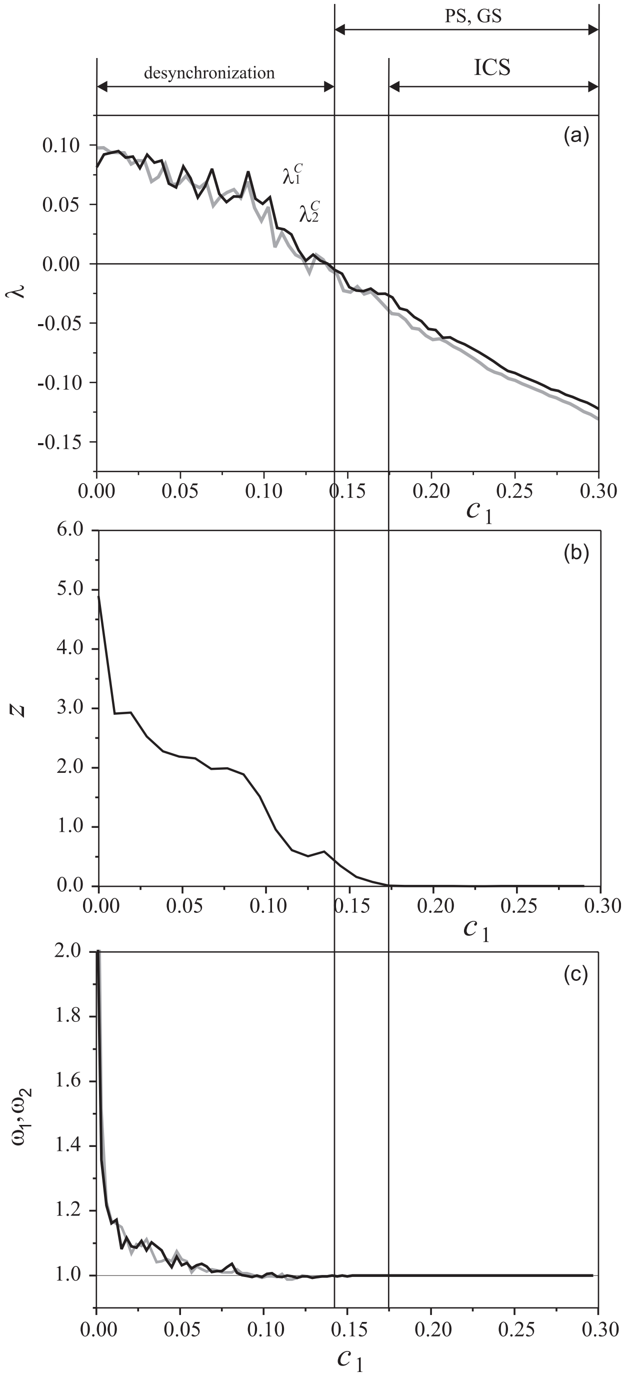

Fig. 6. Plots of the largest RLEs (a), synchronization error z (b) and phase ω_1 and ω_2 (c), versus c_1 for Eq. (8) forced by Eq. (9): $d_1 = 0.0$, $c_2 = 1.0$, $k_1 = 0.1$, $\omega_R = 1.0$, $a_R = 0.15$, $f_R = 0.2$ and $c_R = 10.0$. Regions of desynchronization and CS, PS or GS are indicated. $\Delta = 1\%$ parameter mismatch in c_2 between the first and the second of equation in Eq. (7) has been assumed.

In Fig. 8(a, b) we show the extended analysis of the influence of mismatch in parameters of the system. We calculate the synchronization error z versus c_1 for Eqs. (8) (Fig. 8a) and the threshold of the PS for Eqs. (8) (Fig. 8b). The arbitrary chosen values of mismatch in the range $\Delta \in (0-15\%)$ have been considered. In the frame in Fig. 8a we present the boundary values of ε corresponding to the different values of mismatch Δ. It is

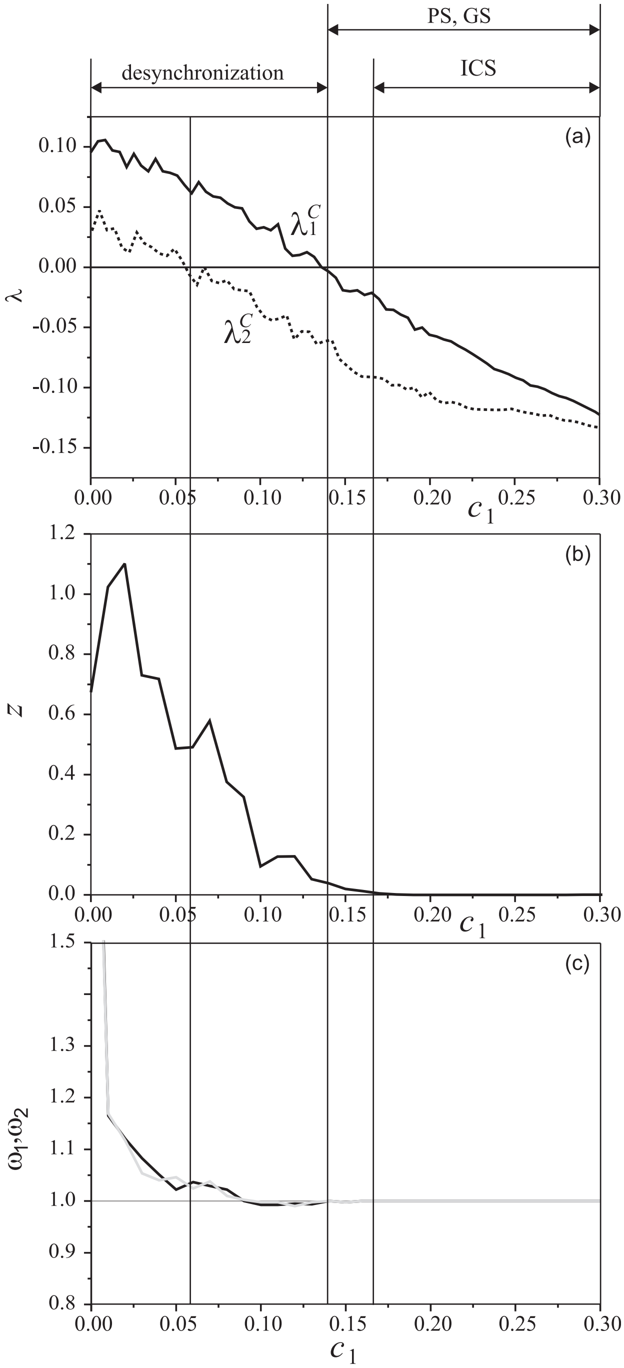

Fig. 7. Plots of the largest RLEs (a), synchronization error z (b) and phase ω_1 and ω_2 (c) versus c_1 for Eq. (8) forced by Eq. (9): $d_1 = 0.08$, $c_2 = 1.0$, $k_1 = 0.1$, $\omega_R = 1.0$, $a_R = 0.15$, $f_R = 0.2$ and $c_R = 10.0$. Regions of desynchronization and CS, PS or GS are indicated. $\Delta = 1\%$ parameter mismatch in c_2 between the first and the second of equation in Eq. (7) has been assumed.

easy to see that with increasing Δ the stable ICS (with ε corresponding to Δ, see Eq. (11)) appears a bit over the PS threshold of parameter c_1.

Thus, we can conclude that the robustness of the ICS is ensured by simultaneous occurrence of the PS.

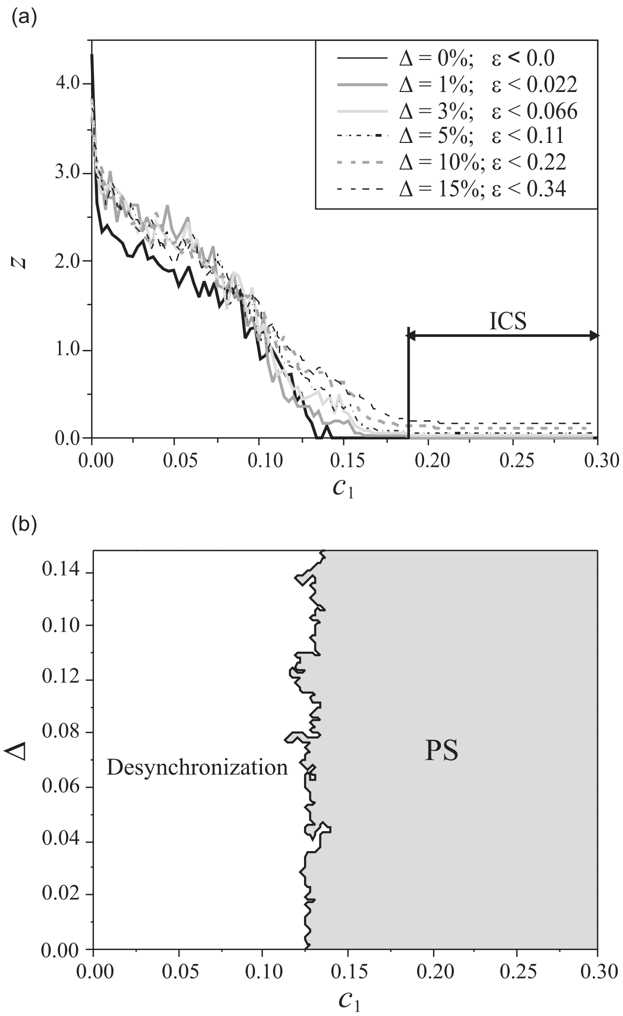

Fig. 8. Plots of the synchronization error z for diffident Δ (a) and threshold of PS (b) versus c_1 for Eq. (8) forced by Eq. (9): $d_1 = 0.0$, $c_2 = 1.0$, $k_1 = 0.1$, $\omega_R = 1.0$, $a_R = 0.15$, $f_R = 0.2$ and $c_R = 10.0$. Regions of desynchronization, ICS and PS are indicated. Δ as parameter mismatch in c_2 between the first and the second of equation in Eq. (7) has been assumed.

5. Conclusions

The analysis presented in this paper demonstrates an existence of the close dependence between the changes in the spectrum of Lyapunov exponents (CLEs and TLEs) and a transition to different types of synchronization. Basing on RLEs, we can predict the behavior of the system. The results of our research show a strict connection between the CS (ICS) of the response oscillators and their PS and GS with the driving system (the (1:1) mode locking)—see Figs. 4–7a, b. The boundary of transmission to the mode locking with the chaotic excitation does not change even in the presence of the small parameter mismatch. Moreover, the CS can also appear under the GS threshold due to a diffusive coupling of identical response subsystems (Fig. 5c). However, a small parameter mismatch destroys such a sensitive synchronous regime and then the ICS occurs only over the GS threshold (Fig. 7c). Thus, the following crucial conclusion results from our studies: a completely collective motion of real oscillators driven externally (i.e., their ICS) is possible only in the case when the GS (or eventually the PS) with the driving occurs. This idea can be extended to other chaotic systems or to large networks with one master oscillator. The reported results seem to be generic as they are preserved for the wide range of system parameters.

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