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Detection and classification of solutions for systems interacting by soft impacts using sample-based method

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In this paper we analyze the dynamics of two externally forced Duffing oscillators interacting via soft impacts using the extended basin stability method. Investigated system is multistable with different types of coexisting solutions (periodic with and without impacts and chaotic). The extended basin stability method let us estimate the occurrence probability of solution or given type of behavior for assumed parameters and initial conditions ranges. We present the general study of system behavior focusing on the existence of a non-impacting periodic solutions. We show how selected controlling parameters and initial conditions affect the response of the system. Proposed method is robust and can be utilize for a variety of engineering applications where the parameters are varying.

Keywords: basin stability, sample based method, soft impact, classification of solutions

1. Introduction

The multistability is a common state in strongly non-linear systems [Czołczyński *et al.*, 2009; Yanchuk *et al.*, 2011; Frazier & Kochmann, 2017; Noël *et al.*, 2017; Galias, 2017; Harne & Goodpaster, 2018]. It results in coexistence of more then one solution for given set of system parameters. Typically, in engineering application we consider multistability as a dangerous state because we are not sure if the system will reach expected solution or it will evolve to different one [Chudzik *et al.*, 2011; Pisarchik & Feudel, 2014; Klinshov *et al.*, 2015; Ruzziconi *et al.*, 2016; Varshney *et al.*, 2018]. Thus, the analysis of coexistence of solutions is crucial for reliability of engineering system.

In case of one degree of freedom (DoF) system we can obtain the knowledge about its dynamics by calculating basins of attraction [H.E. Nusse, 1998; Capecchi & Bishop, 1994; Nusse & Yorke, 1996; de Souza & Caldas, 2001]. We can get the complete information about a number of solutions and size of their basins of attraction. However, in in multistable systems the size of a basin of attraction changes with time evolution along the orbit [Parker Eason *et al.*, 2014; Goncalves *et al.*, 2014]. Recently, we proposed the first measure to characterize the evolution of stability margin along stable periodic orbits [Brzeski *et al.*, 2018]. With the method we are able to describe different sensitivity to a perturbation while without the method one have to calculate several projections of basins of attraction to see how they change. More problems arise when the dimension of the phase space increases, because in such case classical basins of attraction show just a two dimensional projection of multidimensional phase space. Hence, basins of attraction give complete information about dynamics only for one DoF systems. To overcome this problem, one can use one of sample

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based methods. Rega and Lenci proposed "basin integrity measures" [Rega & Lenci, 2005; Lenci & Rega, 2008, 2011; Belardinelli & Lenci, 2016; Belardinelli *et al.*, 2018] that describe the erosion of basins, i.e., global integrity measure and integrity factor. These measures enable to asses if the basin has safe, compact structure or fractal. For low dimensional systems this method gives precise information about structure of phase space. Nevertheless, basin integrity measures are hard to obtain for system with multiple DoF. The second method called "basin stability" has been developed by Menck et. al. [Menck *et al.*, 2013, 2014]. Basin stability method let us to quantify stability basing on the probability of reaching given attractor from random initial conditions. To calculate basin stability measure one has to perform a significant number of Bernoulli trials.Each trial refers to direct numerical integration of the system with randomly chosen initial conditiions. The result of the trial is the type of attractor that is reached. Then, one can calculate the probability of reaching given solution. The idea is that with the sufficient number of trials this probabilty reflects the realtive volume of basin of attraction of given soluton. This method was successfully applied in a numerous applications [Menck *et al.*, 2014; Maslennikov *et al.*, 2014] and different expansions of the mehod were proposed. In our previous papers, we describe extension of this method [Brzeski *et al.*, 2016]. Additionally to initial conditions we draw the parameter values of the system. Such approach let us include in the analysis the uncertainty of parameter values and perform efficient investigation of coexisting solutions for systems with varying parameter values. The proposed method has been validated with experimental study [Brzeski *et al.*, 2017b]. The next method "basin entropy" has been proposed by Daza et. al. [Daza *et al.*, 2016] who propose another sample based algorithm to test the structure of basins of attraction to assess its fractality. The detailed description of sample based methods can be found in a review paper by Brzeski and Perlikowski [Brzeski & Perlikowski, 2018].

In this study, we analyze dynamics of oscillators which interact via soft impacts. Impacts are common in many types of systems, such as gear boxes, tooling machines, vibro-impact oscillators, drilling machines and many more [Lenci & Rega, 1998; Qun-hong & Qi-shao, 2003; Kapitaniak *et al.*, 2018; Liang *et al.*, 2018]. Impacts appear if the motion of an element is limited by a barrier or as an interaction of two elements of a machine. To model such interactions we can employ either hard or soft impact model [Dankowicz & Piiroinen, 2002; Ferrer *et al.*, 2010; Andreaus *et al.*, 2013, 2010]. In the first approach, the time of contact is assumed to be infinitely small and the energy exchange to happen instantaneously, while in the latter the contact time is assumed to have a non-zero value, and the body is assumed to penetrate the base. Hence, interaction via soft impacts requires elastic element in the model of such contact. It is done with a spring (of either the linear or non-linear type e.g. Hertizian) and a viscous damper [Czolczynski *et al.*, 2017; Liu & Chávez, 2017; Liu *et al.*, 2018]. The impacts can be then considered as discontinuous transient coupling, and in this case the equations of such systems motion will have a separate form for in-contact and out-of-contact dynamics. As an interacting system we select the Duffing oscillator because it has nonlinear characteristic and for chosen values of parameters it has two solutions. The coupling introduce additional nonlinearity and increase number of coexisting solutions, which makes such system hard to analyze with classical methods. Thus, considered system is excellent to introduce analysis based on basin stability method.

The paper is organized as follows: in Section 2 we introduce the model which is used to demonstrate the main idea of our considerations, notations used to describe different periodic states of the system and we describe the applied methodology. In the next Section we present the analysis of the influence of different system parameters on response of the system. Finally, in Section 4 conclusions are presented.

2. Model of the system and its periodic solutions

In this section we introduce the model of two Duffing oscillators interacting via discontinuous coupling. Then, we present periodic solutions which exist in the system, their notation and the methodology of calculations performed in this study.

2.1. *Model of the system*

The analyzed system is shown in Fig. 1. It consists of two Duffing oscillators which, while at rest, are separated by a distance d. Both oscillators have the same parameters: bodies of mass M are connected to the wall by a viscous dampers with coefficient c and non-linear springs of hardening type and stiffness

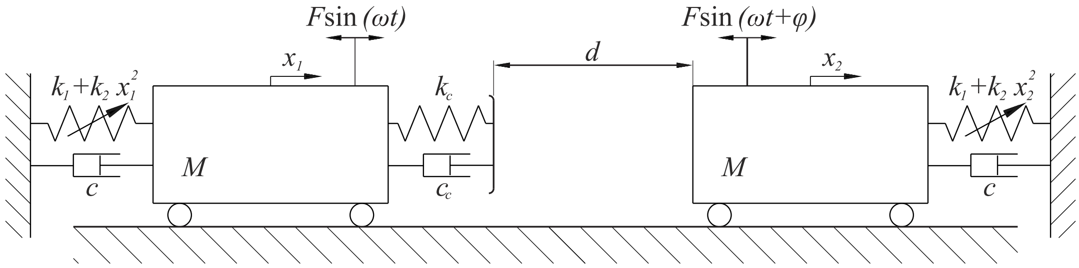

Fig. 1. Model of two Duffing oscillators coupled discontinuously

coefficients $k_1 > 0$ and $k_2 > 0$. The oscillators are driven by two harmonic forces with the same amplitude and frequency (F and ω respectively), but with a phase shift between them. The first oscillator's forcing has fixed phase equal to zero while the phase of second one is equal to φ and its vary in range $\varphi \in [0,\, 2\pi)$.

Depending on the values of d and φ,impacts may occur between the two oscillators. These impacts are of soft type, due to the presence of the spring and damper between the oscillators, with stiffness and damping coefficient equal to k_c and c_c respectively.

The motion of both oscillators is described by the following equations of motion:

$$\begin{aligned} M\ddot{x}_1 + c\dot{x}_1 + k_1 x_1 + k_2 x_1^3 + F_c &= F\sin(\omega t), \\ M\ddot{x}_2 + c\dot{x}_2 + k_1 x_2 + k_2 x_2^3 + F_c &= F\sin(\omega t + \varphi), \end{aligned} \tag{1}$$

where F_c denotes the force resulting from discontinuous coupling of the systems, given by the piecewise formula:

$$F_c = \begin{cases} 0 & \text{for} \quad x_1 - x_2 < d, \\ k_c((x_1 - x_2) - d) + c_c(\dot{x}_1 - \dot{x}_2) & \text{for} \quad x_1 - x_2 \geq d. \end{cases} \tag{2}$$

The values of the system parameters which remain constant in all simulations conducted in this paper are as follows: $M = 1.0\,[kg]$, $k_1 = 1.0\,[\frac{N}{m}]$, $k_2 = 0.01\,[\frac{N}{m^3}]$, $c = 0.05\,[\frac{Ns}{m}]$, $F = 1.0\,[N]$, $\omega = 1.3\,[\frac{1}{s}]$. Parameters d, φ, k_c and c_c are control parameters, and are taken from ranges $d \in [0,\, 22]\,[m]$, $\varphi \in [0,\, 2\pi]\,[rad]$, $k_c \in [5,\, 20]\,[\frac{N}{m}]$, and $c_c \in [1,\, 11]\,[\frac{Ns}{m}]$. Values of the parameters do not directly correspond to real life values, we select them to show the idea of classification of solutions based on the basin stability method using simple model of dynamical system. In order to transform the equations to dimensionless form, we introduce reference length $l_r = 1.0\,[m]$, mass $m_r = 1.0\,[kg]$ and dimensionless time $\tau = t\omega_1$, where $\omega_1 = 1.0\,[\frac{1}{s}]$. Having those values, we can replace dimensional parameters with the following non-dimensional ones: $M' = \frac{M}{m_r}$, $k_1' = \frac{k_1 l_r}{m_r \omega_1^2}$, $k_2' = \frac{k_2 l_r^2}{m_r \omega_1^2}$, $c' = \frac{c}{m_r \omega_1}$, $F' = \frac{F}{m_r l_r \omega_1^2}$, $\omega' = \frac{\omega}{\omega_1}$, $k_c' = \frac{k_c l_r}{m_r \omega_1^2}$, $c_c' = \frac{c_c}{m_r \omega_1}$, $d' = \frac{d}{l_r}$. For simplicity all primes will be omitted in the analysis.

2.2. *Periodic solutions*

In this paper we show the continuation of our previous studies on the dynamics of presented system [Brzeski *et al.*, 2017a; Chávez *et al.*, 2017]. Previously, we investigated periodic solutions and their bifurcations. We found out that with decrease of the distance d between Duffing systems the number of existing solutions is reduced. Now, let us briefly recall the results from the aforementioned papers to describe the dynamics of considered system. The single Duffing oscillator for selected parameter values has three periodic solutions due to its hardening characteristic. Two of them are stable and one is unstable, hence we just focus on two stable orbits with small and large amplitude respectively. When we consider two interacting systems there are four possible states, i.e., the first and the second system are both on small (solution No. I) or large

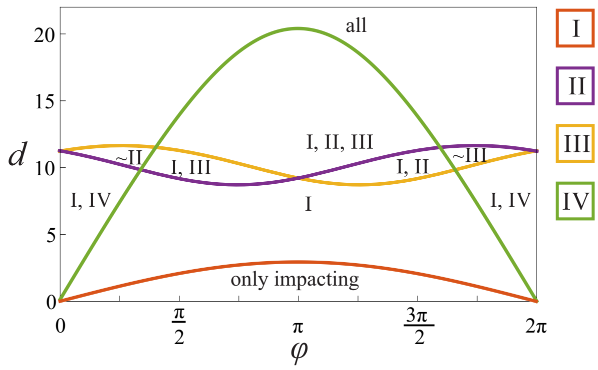

Fig. 2. Border lines of stability for non-impacting solutions in the $(d,\ \varphi)$ plane (a). Each type of solution is drawn with line of different color. ($M = 1.0$, $k_1 = 1.0$, $k_2 = 0.01$, $c = 0.05$, $F = 1.0$, $\omega = 1.3$).

(solution No. IV) amplitude attractors and the first is on small while the second on large solution (solution No. II) and vice versa (solution No. III).

The current distance between the oscillators can be measured as $x_1 - x_2 - d$ and it depends on the relative position of each oscillator along the trajectory governed by the phase shift φ. Impact between the two oscillators occurs when the distance $x_1 - x_2 - d$ is zero, while in contact distance is negative. In Figure 2 two parameter plot is presented showing the boundaries below which given aforementioned four periodic solutions without impacts exist. The boundary line of each solution is marked by different color. Below the red line on the graph Solution No. I is not possible despite the values of d and φ. Consequently, other lines depict borders of existence of the remaining periodic solutions which disappear when d and φ are below the lines (details of calculation of this diagram are presented in our previous paper [Brzeski *et al.*, 2017a]). These lines give only information about non-impacting periodic solutions but we have no data about solutions with impacts. In this study we investigate this problem. We select three values of parameter d (5, 10.5 and 15) to investigate the dynamics of system for three significantly different states (number of non-impacting solutions vary from one to four).

2.3. *Extended basin stability method*

The idea of extended basin stability method has been described in our previous papers [Brzeski *et al.*, 2016, 2017b; Brzeski & Perlikowski, 2018]. It bases on basin stability method introduced by Menck et. al. [Menck *et al.*, 2013, 2014]. They proposed a powerful tool to estimate the size of complex basins of attraction in multidimensional systems which in case of system with discontinuity is very useful. To determine the structure of phase space we perform N number of Bernoulli trials of direct numerical integration. Hence, we have to draw N sets initial conditions from assumed ranges of accessible initial conditions for fixed parameters of the system (in all dynamical system we have limitation of accessible amplitude or velocity, so we have to take those limits into account). Then, for each set of initial values we check the type of final attractor. We integrate system with Runge-Kutta 4th order method with constant time step equal to $5.0e^{-4}[-]$. To precisely detect moments of contact and separation (switching of system equations) we monitor state of the system and close to both events we decrease the size of the time step ten times. Based on this the percentage distribution of solutions is determined.

In [Brzeski *et al.*, 2016; Brzeski & Perlikowski, 2018] we extend the above method with assuming that parameters of the system are also drawn for each trial. This let us include in the algorithm uncertainties and mismatches of parameters or investigate the system with varying parameters (e.g. frequency of excitation). We can also use this method during design of system to find parameters ranges where its dynamics is most suitable for given application.

General equation of motion of dynamical system can be written in following form: $\dot{x} = f(x, \gamma)$, where $x \in \mathbb{R}^n$ is state vector and $\gamma \in \mathbb{R}^m$ is parameters vector. Let $B \subset \mathbb{R}^n$ be a set of accessible initial conditions and $C \subset \mathbb{R}^m$ a set of the system parameters values. Let us assume, that an attractor $\mathcal{A}$ exists for $\gamma \in C_{\mathcal{A}} \subset C$

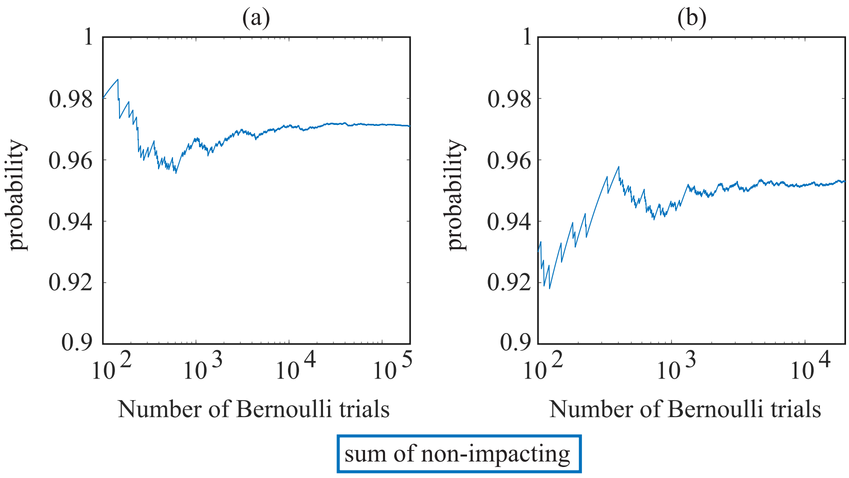

Fig. 3. Convergence of probability of reaching a non-impacting solutions for as a function of number of Bernoulli trials (logarithmic scale) for $d = 5$. In panel (a) we show convergence for free phase φ (up to $N = 200000$ trials) and panel (b) for fixed phase φ (up to $N = 20000$ trials).

and has a basin of attraction $\beta(\mathcal{A})$. Assuming random initial conditions from the set B the probability that the system will reach attractor $\mathcal{A}$ is given by $p(\mathcal{A})$. If this probability is equal to $p(\mathcal{A}) = 1.0$ this means that the given solutions is the only one in the taken range of initial conditions and given values of the parameters. Otherwise, other attractors coexist.

In this study, we focus on application of the proposed extended method to design the system behaving in prescribed way, i.e, we want to avoid solutions with impacts by selecting parameters of the stop (k_c, c_c) and phase shift (φ) for selected values of distance d between the two interacting Duffing systems.

3. Results

In this section we show results obtained for system of two interacting Duffing oscillators. We investigate the probability of reaching periodic, non-impacting solutions as a function of system parameters and initial conditions. We draw parameters of system from following ranges: $c_c \in [1, 11]$, $k_c \in [5, 20]$ and $\varphi = [0,\ 2\pi)$ (in one case we fix φ) and initial conditions $(x_{1,2},\ \dot{x}_{1,2})$ are from range: $[-12,\ 12]$ for three fixed values of distance d. Generally, ranges of initial conditions should be select individually for each system to correspond to maximum and minimum values reached be system trajectories or specific properties of system (constrains on amplitude or/and velocity). We perform $N = 200000$ Bernoulli trials in case when we draw all parameters and $N = 20000$ Bernoulli trials for fixed phase shift φ. To validate that the assumed numbers of Bernoulli trials are correct we performed convergence study. In Fig. 3 we show how probability of reaching non-impacting solutions for distance $d = 5$ is varying as a function of trials numbers. In both figures number of Bernoulli trials is given in logarithmic scale. Panel (a) shows plot for up to $N = 200000$ trials (free phase φ) and panel (b) for up to $N = 20000$ trials (fixed phase φ). In both cases we see the convergence of probabilities, hence we claim that such choice of trials numbers is sufficient to obtain reliable results.

We show that extended basin stability method let us obtain the information about types of solutions in the system. Moreover, it is an effective tool to predict dynamics of the analyzed strongly non-linear system.

3.1. *Influence of coupling parameters c_c and k_c on the presence of non-impacting solutions for fixed φ*

We start the analysis with study of influence of stop parameters (k_c and c_c) on the probability of presence of non-impacting solutions. In the first case, to simplify analysis we fix the value of the phase shift to $\varphi = 5.28$. Results are presented in Fig. 4 for three values of distance d between systems, i.e., $d = 5$ (Fig. 4(a)), $d = 10.5$ (Fig. 4(b)) and $d = 15$ (Fig. 4(c)). We divide the parameters space (k_c, c_c) into 225 equally spaced squares and calculate the probability of reaching non-impacting periodic solutions (sum of probability of solutions

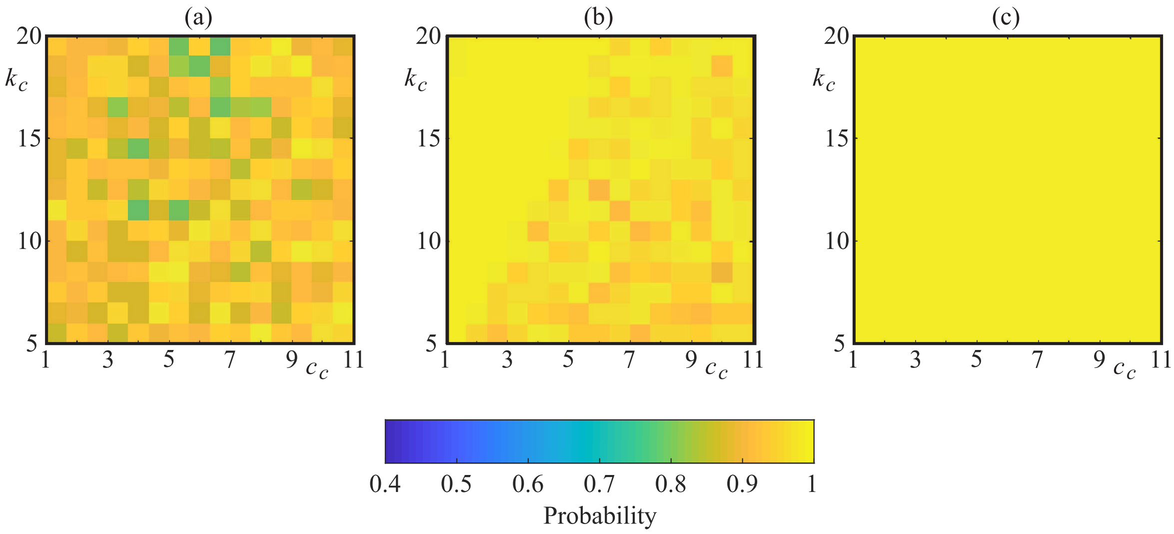

Fig. 4. The probability of reaching a non-impacting solution with respect to values of the coupling parameters k_c and c_c for fixed $\varphi = 5.28$ and $d = 5$ (a), $d = 10.5$ (b) and $d = 15$ (c). Calculations carried out for the following system parameters: $M = 1.0$, $k_1 = 1.0$, $k_2 = 0.01$, $c = 0.05$, $F = 1.0$, $\omega = 1.3$.

Nos I-IV) in each square. The maximum value is 1.0 which indicates that in given range we do not observe solutions with impacts. As we can see in panel (a) the probability is varying from 0.79 to 0.99 and its mean value is 0.90 so generally it is high, however it strongly vary between selected squares. When we increase the distance to $d = 10.5$ we see that the minimum value of probability is equal to 0.89, maximum to 1.0 and mean to 0.96. Contrary to smaller distance between Duffing system (subplot (a)), now we can identify the large region with only non-impacting periodic solutions in upper left part of the diagram. We also observe increase of the mean probability value. In the last panel we see that for all drawn values of system parameters and initial conditions the system always reach periodic non-impacting solutions. Hence, we are certain that we always achieve one of the non-impacting solutions (Nos I-IV).

3.2. *Influence of coupling parameters c_c and k_c and the phase shift φ on the presence of non-impacting solutions*

In this subsection we extend previous study with drawing also value of the phase shift φ. We present results in Fig. 5(a-c) for the same values of distance d as in previous subsection. We also divide the parameters space (k_c, c_c) into 225 equally spaced squares and calculate the probability of reaching non-impacting periodic solutions (sum of probability of solutions Nos I-IV) in each square. Now, we see that changes of φ influences dynamics of the system significantly. In panel (a) the minimum value of probability is equal to 0.44, mean to 0.92 and maximum to 0.98, thus we see that random phase shift is decreasing minimum value of probability but increase its mean value. In panel (b) for distance $d = 10.5$ we see decrease of probability (minimum: 0.83, mean: 0.87 and maximum: 0.93). The probability is higher for larger values of k_c and smaller c_c. For all values of k_c the increase of damping c_c causes drop down of probability. Moreover, we do not have the range where impacting solution are not present. In the last case (panel (c), $d = 15.0$) the values of probabilities are as follows: minimum: 0.92, mean: 0.97 and maximum: 0.99. Thus, the probability is overall high and it increases with the increase of damping coefficient c_c. Generally, we see that one can find ranges of high probability when all parameters and initial conditions are drawn, however we observe significant influence of variable phase shift. Hence, to achieve the range with probability equal to one we have to fix the value of φ or draw it from smaller range then in this case.

3.3. *Influence of phase shift on the presence of non-impacting solutions*

As we conclude in previous subsection phase shift is important parameter. Now, we focus on its influence on probability of reaching non-impacting solutions. We take coupling parameters from ranges $c_c \in [1, 11]$

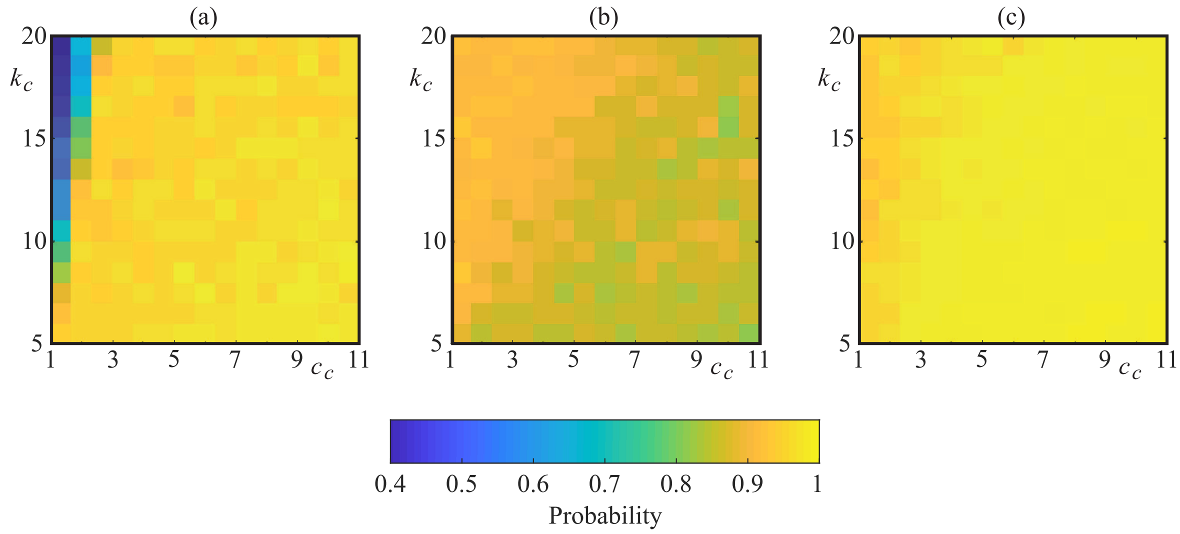

Fig. 5. The probability of reaching a non-impacting solution with respect to the coupling parameters k_c and c_c with varying $\varphi = [0, 2\pi)$ and $d = 5$ (a), $d = 10.5$ (b) and $d = 15$ (c). Calculations conducted for the following system parameters: $M = 1.0$, $k_1 = 1.0$, $k_2 = 0.01$, $c = 0.05$, $F = 1.0$, $\omega = 1.3$.

and $k_c \in [5, 20]$, hence we are able to determine ranges of phase shift $\varphi = [0, 2\pi)$ where only non-impacting solutions exist. Results are presented in Figure 6. Similarly to previous study we investigate three different values of distance between Duffing oscillators: $d = 5$ (a), $d = 10.5$ (b) and $d = 15$ (c). We divide range of φ into 32 equal sets. For each set we calculate the probability of reaching four non-impacting periodic solutions (Nos I - IV) and additionally we plot their sum (see description in subsection 2.2). Probabilities of solutions are marked with squares in the middle of each set with colors corresponding to given solution. Lines connecting squares show tendency. Blue color squares indicate the sum of non-impacting solutions, hence when probability reaches 1.0 it means that there are only non-impacting solutions. With the increase of distance d the number of coexisting solutions also increases according to results shown in Figure 2. There is no symmetry in the probability with respect to $\varphi = \pi$ but for a such non-linear system it is not surprising, because the second system is not a mirror refection of the first one due to phase shift φ. For small value of distance $d = 5.0$ we see that only non-impacting orbits are present for range where two solutions coexist (Nos I and IV). When solution No. IV destabilizes we see coexistence of solution No. I and impacting attractors. For distance $d = 10.5$ according to Figure 2 we observe all non-impacting solutions but they never all coexist for fixed phase shift .The sum of their probabilities of occurrence is equal 1.0 only close to $\varphi = \pi$. For panel (c) obtained for distance $d = 15.0$ we observe increase of the range where the sum of probabilities is equal to unity. Nevertheless, for all three cases the probability of non-impacting solutions strongly depends on the phase shift, hence we are not certain that with arbitrary, fixed values of stop (assuming$c_c \in [1, 11]$ and $k_c \in [5, 20]$) and phase shift $\varphi = [0, 2\pi)$ system will behave according to the assumption (only non-impacting solutions). Thus, the conclusion from the previous subsection is confirmed.

3.4. *Influence of initial conditions on the presence of non-impacting solutions*

The results of influence of initial conditions are presented in Figure 7. In panels (a-c) we show the probability of reaching periodic solutions as a function of initial displacements (x_1 and x_2) for three values of distance equal to $d = 5$, $d = 10.5$ and $d = 15$ respectively. The red lines indicate the impact condition given by Eq. (2), hence above this line Duffing oscillators at initial state are in contact. While in panels (d-f), we present the probability of reaching periodic solutions as a function of initial velocities ($\dot{x}_1$ and $\dot{x}_2$) for the same values of distance d ($d = 5$, $d = 10.5$ and $d = 15$). The values of parameters (k_c, c_c and φ) and second pair of initial conditions are random.

Let us start the analysis from the first row. In panel (a) we see that for majority of two parameters range the probability is high and only in the small range in the top right corner the probability is rapidly

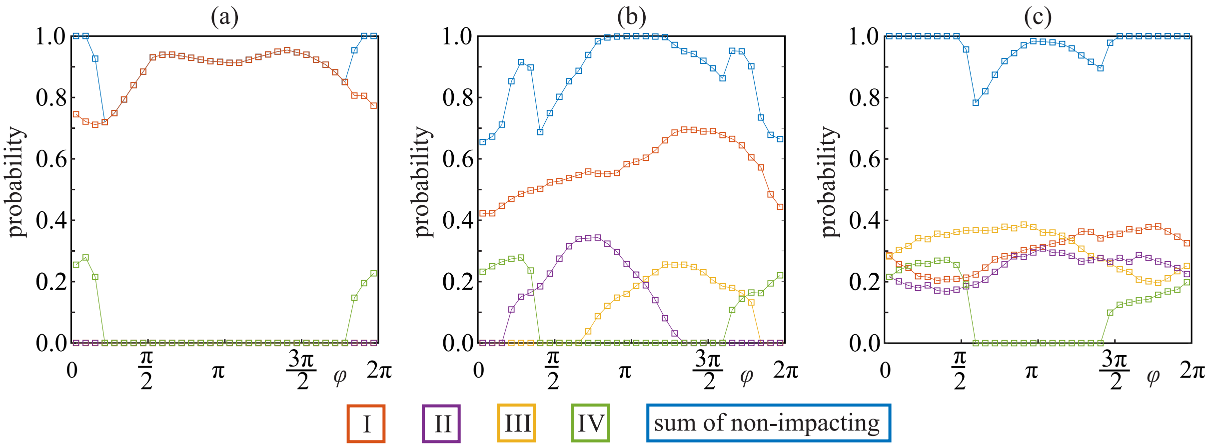

Fig. 6. The probability of reaching a non-impacting solution as a function of $\varphi = [0,\, 2\pi)$ and $d = 5.0$ (a), $d = 10.5$ (b) and $d = 15.0$ (c) for $c_c \in [1, 11]$ and $k_c \in [5, 20]$. Calculations conducted for the following system parameters: $M = 1.0$, $k_1 = 1.0$, $k_2 = 0.01$, $c = 0.05$, $F = 1.0$, $\omega = 1.3$.

dropping down to minimum value equal to 0.54. If we exclude this low probability region the mean value of probability is equal to 0.98. For $d = 10.5$ (panel (b)) we see significant difference in comparison to the previous plot. Probability is low for larger range of initial conditions values (mean value is equal to 0.86). The higher probability occurs only close to minimum values of the first initial condition ($x_1 \approx -12$). Much bigger probability values are obtained for the largest distance $d = 15$. Here, minimum value is equal to 0.87 and mean to 0.96. We see the large range of probability equal to 1.0 in center and for low values of the first initial condition. Thus, in this range the influence of stop parameters, phase shift and initial velocities is small.

Now, we focus on the influence of initial velocities. In panel (d) we see that the probability is varying from 0.71 to 0.97 with mean value 0.93. We see that low and high probability ranges are sharply divided. With the increase of the distance d we see that the range with low probability increases. Hence, we observe that the mean value is also lower and equal to: 0.89. Similarly as for initial displacement, for distance $d = 15$ there is a wide range with probability equal to unity (the mean value is equal to 0.97 and minimum to 0.88). Thus, taking initial conditions from this range, we are certain that non-impacting solution is reached.

4. Conclusions

In this paper we show a method to analyze the dynamics of complex non-linear, non-smooth systems. We use sample based approach originate from the extended basin stability method. As a random parameters we take the parameters describing soft impacts between the two Duffing oscillators. Namely, the stiffness of the spring and damping coefficient of dash-pot. Apart from that we analyze the influence of the phase shift of excitation of the second Duffing oscillator. In the last part we also show the influence of four initial conditions of the Duffing systems.

Presented method can be utilized to better understand system dynamics or to analyze the occurrence of some specific behavior of the system. In the paper we investigate the probability of reaching non-impacting, periodic solutions, but one can select any goal of analysis, e.g, chaotic solutions, solutions with assumed amplitude or other. We can also simulate the real-life uncertainty of the parameter values, variability of parameters during operation or slow change of their values due to the fatigue of the device. Thus, we can define the ranges of the parameters space in which the considered system operates correctly and describe how change in parameter values (and/or initial conditions) influence its behavior. We are able to estimate the risk that the system will behave differently from the expected way. The proposed method is robust, and can be utilized for a wide variety of engineering applications. Among them we can mention following examples of multistable systems with impacts: vibro-impact drilling systems [Liu *et al.*, 2018; Liao *et al.*, 2018], cutting machines [Yan *et al.*, 2017], multilevel DC/DC converters [Zhusubaliyev *et al.*, 2015], piecewise smooth rotor/stator rubbing systems [Li *et al.*, 2017], Filippov-type systems [Glendinning *et al.*, 2016], systems

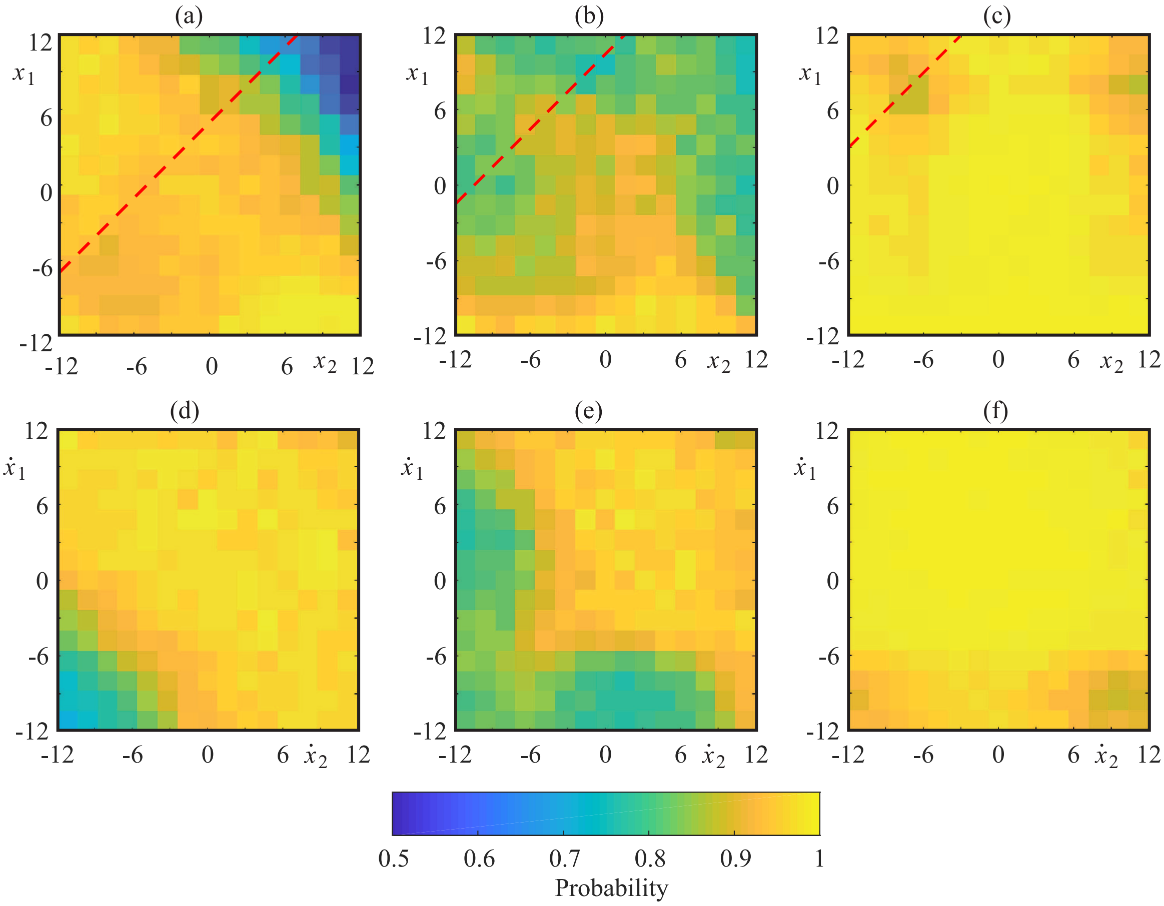

Fig. 7. The probability of reaching a non-impacting solution as a function of initial conditions and $d = 5.0$ (a), $d = 10.5$ (b) and $d = 15.0$ (c) for of $\varphi = [0,\ 2\pi)$, $c_c \in [1, 11]$ and $k_c \in [5, 20]$. The calculations were conducted for the following system parameters: $M = 1.0$, $k_1 = 1.0$, $k_2 = 0.01$, $c = 0.05$, $F = 1.0$, $\omega = 1.3$.

with a frictional unilateral constraint [Lancioni *et al.*, 2009].

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