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Huygens' odd sympathy experiment revisited

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Abstract: We repeat Huygens' experiment using real pendulum clocks in the same way as it was done originally, i.e., we hang two clocks on the same beam and observe the behavior of the pendulums. The clocks in the experiment have been selected in such a way to be as identical as possible. It has been observed that when the beam is allowed to move horizontally, the clocks can synchronize both in-phase and anti-phase. We perform computer simulations of the clocks' behavior to answer the question how the nonidentity of the clocks influences the synchronization process. We show that even the clocks with significantly different periods of oscillations can synchronize, but their periods are modified by the beam motion so they are no more accurate.

Keywords: Pendulum, clocks, synchronization, Huygens' experiment

1. Introduction

The phrase **odd sympathy** (or exactly the **odd kind of sympathy**) was used by Dutch physicist Christiaan Huygens (1629-95) in a letter to the Royal Society of London pertaining to the tendency of two pendulums to synchronize, or anti-synchronize, when mounted together on the same beam [Huygens, 1665]. The original drawing showing this experiment is shown in Fig. 1. Huygens, the inventor of the pendulum clock [Huygens, 1673], noticed the effect while lying in bed. Two pendulum clocks, mounted together, will always end up swinging in exactly opposite directions, regardless of their respective individual motion. This was one of the first observations of the phenomenon of coupled harmonic oscillators, which have many applications in physics [Pikovsky et al., 2001; Blekham, 1988]. Huygens originally believed the synchronization occurs due to air currents shared between the two pendulums, but later after performing several simple tests he dismissed this idea and attributed sympathetic motion of pendulums to imperceptible movement in the beam from which both pendulums are suspended.

Recently, this idea has been rediscussed by a few groups of researchers which tested Huygens' idea

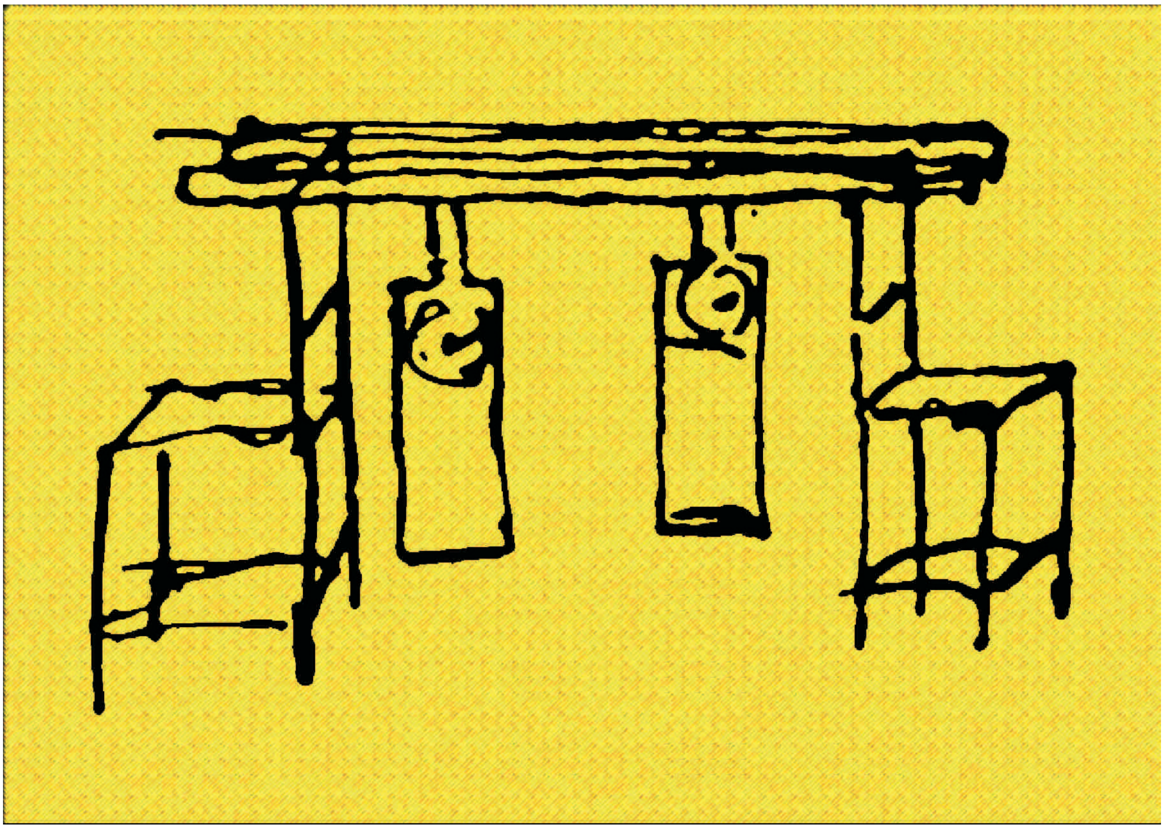

Fig. 1: An original drawing describing the Huygens' experiment [Pikovsky et al., 2001; Huygens, 1665].

[Pogromsky et al., 2001; Bennet et al., 2002; Senator, 2006; Dilao, 2009; Kumon et al., 2002; Fradkov, & Andrievsky, 2007; Pantaleone, 2002; Ulrichs et al., 2009; Czolczynski et al., 2009b]. To explain Huygens' observations Bennett et al. [2002] built an experimental device consisting of two interacting pendulum clocks hanged on a heavy support which was mounted on a low-friction wheeled cart. The device moves by the action of the reaction forces generated by the swing of two pendulums and the interaction of the clocks occurs due to the motion of the clocks' base. It has been shown that to repeat Huygens' results, the high precision (the precision that Huygens certainly could not achieve) is necessary.

The system very close to one considered by Huygens (i.e., two pendulum clocks with cases hanging from the same beam) has been investigated by Senator [2006] who developed a qualitative approximate theory of clocks' synchronization. This theory explicitly includes the essential nonlinear elements of Huygens' system, i.e., escapement mechanisms but also includes many simplifications.

A device mimicking Huygens' clock experiment, the so-called "coupled pendulums of the Kumamoto University" [Kumon et al., 2002], consists of two pendula which suspension rods are connected by a weak spring, and one of the pendulums is excited by an external rotor. The numerical results of Fradkov and

Andrievsky [2007] show simultaneous approximate in-phase and anti-phase synchronization. Both types of synchronization can be obtained for different initial conditions. Additionally, it has been shown that for small difference in the pendulums frequencies they may not synchronize.

The problems of clocks synchronization is also Blekhman [1988] where the clocks have been modeled as van-der Pol's type self-excited oscillators.

In this paper we repeat the Huygens' experiment using real pendulum clocks. We have been trying to perform the experiment in the same way as Huygens did it. We hang two clock on the same beam and observe the behavior of the pendulums. The clocks in experiment have been selected in such a way to be as identical as possible. It has been observed that when the beam is allowed to move horizontally the clocks can synchronize both in-phase and anti-phase. As we notice some small differences in the pendulum lengths and periods (so small to be identified in the Huygens' time) we perform computer simulations to answer the question: how the nonidentity of the clocks influences the synchronization process. We show that even the clocks with significantly different periods of oscillations can synchronize, but their periods are modified by the beam motion so they are obviously no more accurate.

This paper is organized as follows. In Sec. 2 we have described the results of our experiments in which we used contemporary mechanical pendulum clocks. Sec. 3 presents the results of our numerical simulations. It describes the model of the clocks which has been used and answer the question how the non-identity of the clocks influences the synchronization process. Finally, we summarize our results in Sec.4.

2. Experimental observations

For our experiments we take two contemporary pendulum clocks (type: SN–13, produced in 2003 in Factory of Clocks in Torun, Poland) which are shown in Fig. 2(a-c). These clocks have typical escapement mechanisms described in [Rowlings, 1944; Czolczynski et al., 2009a] and pendulums of the length 0.269 [m] and mass 0.158 [kg]. The total mass of the clock is equal to 5.361 [kg]. The clocks are placed in the wooden cases. The clocks in experiment have been selected in such a way to be as identical as possible but we noticed a small time difference of 1 [s] after 24 hours. When the clocks are hanging on the wall as in Fig. 2(a) no synchronization has been observed.

Next the clock has been hanged on the wooden beam (length - 1.13 [m], mass - 1.45 [kg]) and located on two chairs as in the original Huygens experiment of Fig. 1. Our setup is shown in Fig. 2(b). In this case

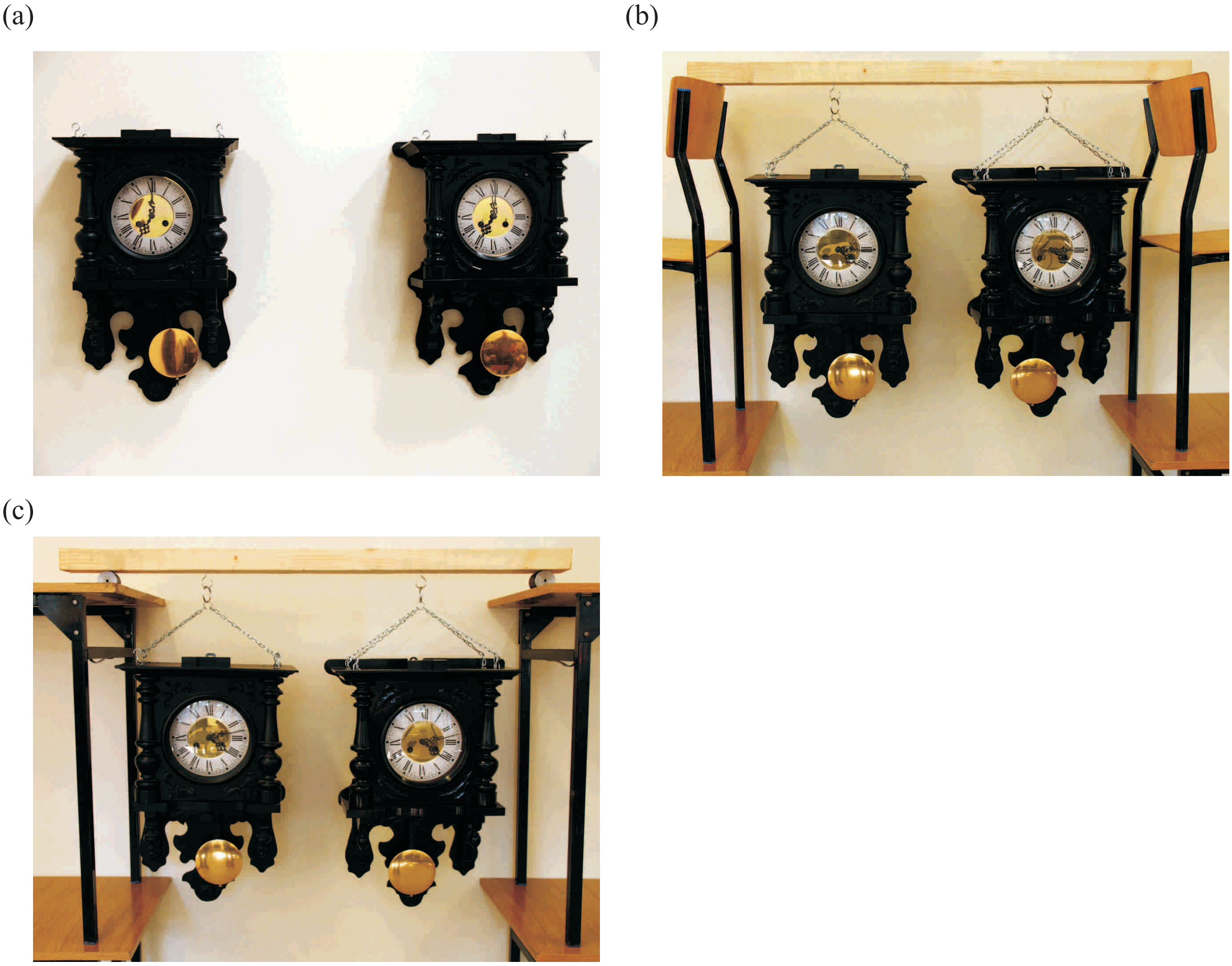

Fig. 2: Experiments with two pendulum clocks: (a) clocks hang on the wall, (b) clocks hang from the beam located on the chairs' backs, (c) clocks hang from the beam which can roll horizontally on the tables' desks.

we have not observed the clocks' synchronizations but we noticed the frequent switches off of one of the clocks (amplitude death of its pendulum). This effect occurs due to the spherical motion of the clocks' cases and subsequently their pendulums. The spherical motion of the pendulums (with the too large amplitudes) switches off the escapement mechanism which is designed for the planar motion of the pendulum. To reduce the amplitudes of the spherical oscillations we have balanced the clocks by adding additional masses into their cases but we have not observed the synchronization.

The synchronization has been observed in the setup shown in Fig. 2(c). Two chairs have been replaced by the tables with a flat horizontal desks. The beam with hanging clocks has been located on the rolls which can roll on the table desks. We have been trying to reduce friction by polishing the surfaces of the beam, table desks and rolls. Depending on initial conditions it has been possible to observe both in-phase

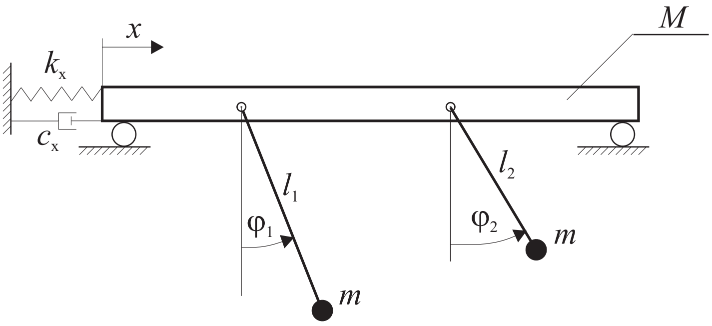

Fig. 3: The model of the system.

and anti-phase synchronization of clocks' pendulums.

In -phase and anti-phase synchronization can be also observed in the case when the clocks' cases are mounted to the beam (the relative case-beam motion is suppressed). However, this case is closer to Bennet et al. [2002] experiment than to original Huygens one.

3. Numerical simulations

3.1. *Model*

The analyzed system is shown in Fig. 3, it consists of the rigid beam and two clocks suspended on it. The beam can move in a horizontal direction, its movement is described by coordinate x. The mass of the beam is connected to the refuge of a linear spring and linear damper k_x and c_x. Clocks' pendulums have the same mass m but different lengths l_1 and l_2. The motion of the pendulums is described by angles φ_1 and φ_2 and is damped by dampers (not shown in Fig. 1) with damping coefficients $c_{\varphi 1}$ and $c_{\varphi 2}$. The pendulums are driven by the escapement mechanism described in details in [Rowlings, 1944; Czolczynski et al., 2009a]. Notice that when the swinging pendulums do not exceed a certain angle γ_N the escapement mechanisms generate the constant moments M_{N1} and M_{N2}. This mechanism acts in two successive steps (the first step is followed by the second one and the second one by the first one). In the first step if $0 < \varphi_i < \gamma_N$ $(i = 1, 2)$ then $M_i = M_{Ni}$ and when $\varphi_i < 0$ then $M_{Ni} = 0$. For the second stage one has for $-\gamma_N < \varphi_i < 0$ $M_i = -M_{Ni}$ and for $\varphi_i > 0$ $M_{Ni} = 0$.

The equations of motion are as follows:

$$\begin{aligned} &ml_i^2\ddot{\varphi}_i + ml_i\ddot{x}\cos\varphi_i + c_{\varphi i}\dot{\varphi}_i + mgl_i\sin\varphi_i = M_i \quad i = 1, 2, \\ &(M + 2m)\,\ddot{x} + c_x\dot{x} + k_x x + \textstyle\sum_{i=1}^{2}\left(ml_i\ddot{\varphi}_i\cos\varphi_i - ml_i\dot{\varphi}_i^2\sin\varphi_i\right) = 0. \end{aligned} \tag{1}$$

All the parameters of Eq. (1) are dimensionless. One should distinguish between three periods of pendulums' oscillations; (i) the period of the pendulums' oscillations in the case when the beam is at rest

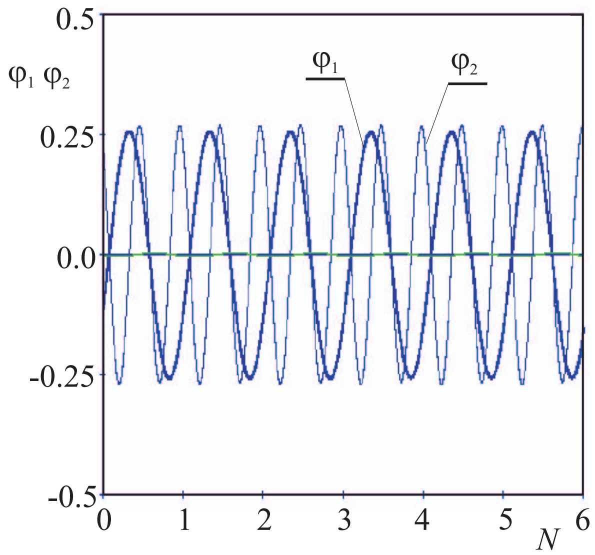

Fig. 4: Oscillations of two pendulum; scale of pendulum 2: $\xi = 0.5$.

denoted by T_{i0}, $(i = 1, 2)$, (ii) the periods of the pendulums oscillating on the moving beam (i.e., the time in which the pendulum reaches the maximum deflection and is moving in the same direction) denoted by T_i, (iii) the period of the oscillations of the beam-pendulums system denoted by T. Under these assumptions the dynamics of the pendulum clock is described by a self-excited oscillator with a limit cycle [Andronov, 1966] (see also [Moon & Stiefel, 2006]). The dynamics of the other type of the clock escapement mechanism, i.e., verge and foliot mechanism is described in [Rowlings, 1944; Roup et al., 2003; Lepschy et al., 1993].

In our numerical calculations we consider the following parameter values. The first pendulum is characterized by: $m = 1.0$, $l_1 = g/4\pi^2 = 0.2485$ (i.e., in the case when the beam is at rest its period of limit cycle oscillations is equal to $T_0 = 1$ and its frequency is equal to $\alpha_{10} = 2\pi$; $g = 9.81$), $c_{\varphi 1} = 0.01$, $M_{N1} = 0.075$, $\gamma_N = 5°$ (i.e., when the beam is at rest the pendulum performs the oscillations with amplitude $\Phi_{10} = 15°$). The second pendulum has the same mass m and γ_N Its length is equal to $l_2 = \xi^2 l_1$ (i.e., $\alpha_2 = \alpha_1/\xi$ and $T_{20} = \xi T_{10}$) and $c_{\varphi 2} = \xi c_{\varphi 1}$, $M_{N2} = \xi M_{N1}$. The coefficient ξ is called the scale factor of the second pendulum. Proportionality of the damping coefficients and escapement mechanisms moments of both pendulums results in the equality of the amplitudes $\Phi_{20} = \Phi_{10}$. In Fig. 4 we show the oscillations of pendulums 1 (thick line) and 2 (thin line) suspended on the nonmoving beam. The displacements of both pendulums φ_1 and φ_2 versus time represented by the number of oscillation periods of the first pendulum N are presented. As $\xi = 0.5$ one can observe that $T_{20} = 0.5T_{10}$ and that the amplitudes of both pendulums are the same. Beam's parameters are as follows. Its mass is assumed to be $M = 5.0$, the stiffness k_x and damping coefficients c_x are related to the beam mass, i.e., $k_x = 0.1M$, $c_x = M$.

3.2. *Different types of synchronization*

Fig. 5(a-c) and Fig. 6(a-c) show respectively the bifurcation diagrams of the system (1) calculated for decreasing and increasing values of ξ . In Fig. 7(a-f) the time series of the characteristic system behavior are presented. In bifurcation diagrams the position of the pendulum 2 is registered at the moment when the pendulum 1 passes through 0 with a positive velocity ($\varphi_1 = 0.0$ $\dot{\varphi}_1 > 0.0$) is plotted versus ξ parameter. The calculations presented in Fig. 5(a) started at $\xi = 1.0$ (i.e., identical pendulums) and the following initial conditions $x_0 = 0.0$, $\dot{x}_0 = 0.0$, $\varphi_{10} = 0.26$, $\dot{\varphi}_{10} = 0.0$, $\varphi_{20} = 0.26$, $\dot{\varphi}_{20} = 0.0$ for which the system reaches a state of in-phase synchronization shown in Fig. 7(a). Next, the value of has been reduced by the step $\Delta\xi = 0.00125$ up to $\xi = 0.5$. With the decrease of ξ (i.e., the shortening of the length of pendulum 2), in-phase synchronization persists till $\xi = 0.92$. Fig. 7(a) presents the example of 1:1 in-phase synchronization for $\xi = 0.95$ and the following initial conditions ($x_0 = 0.0$, $\dot{x}_0 = 0.0$, $\varphi_{10} = 0.26$, $\dot{\varphi}_{10} = 0.0$, $\varphi_{20} = 0.26$, $\dot{\varphi}_{20} = 0.0$). Both pendulums are moving in anti-phase to the motion of the beam so their periods of oscillations decrease $T_1 < T_{10}$ and $T_2 < T_{20}$. The differences of the values of the amplitudes of both pendulums cause that $T_{10} - T_1 > T_{20} - T_2$ and after the transient the systems reach common value $T = 0.84$.

For the smaller values of ξ one observes the transition to anti-phase synchronization shown in Fig. 7(b). This type of behavior is preserved up to $\xi = 0.55$. Notice that for $\xi = 0.55$ the ratio of the lengths of both pendulums is equal to $l_2/l_1 = \xi^2 = 0.3025$ (i.e., the triple difference in the lengths). In Fig. 7(b) an example of anti-phase 1:1 synchronization obtained for $x_0 = 0.0$, $\dot{x}_0 = 0.0$, $\varphi_{10} = 0.26$, $\dot{\varphi}_{10} = 0.0$, $\varphi_{20} = -0.26$, $\dot{\varphi}_{20} = 0.0$ is shown. Pendulum 1 moves in anti-phase with the motion of the beam so its period of oscillations decreases ($T_1 < T_{10}$) while pendulum 2 moves in phase with the beam motion and its period increases ($T_2 > T_{20}$). After an initial transient the periods of oscillations reach the common value $T_{20} < T = 0.98 < T_{10}$

With further increase of the values of ξ in the interval $0.55 > \xi > 0.51$ one observes the quasiperiodic behavior. In this interval the desynchronization of both pendulums occurs. The example of quasiperiodic behavior for $\xi = 0.6871$ is shown in Fig. 7(c,d). Time series of the system (1) are shown in Fig. 7(c) and the Poincare map showing the position of pendulum 2 at the time when pendulum 1 is moving through

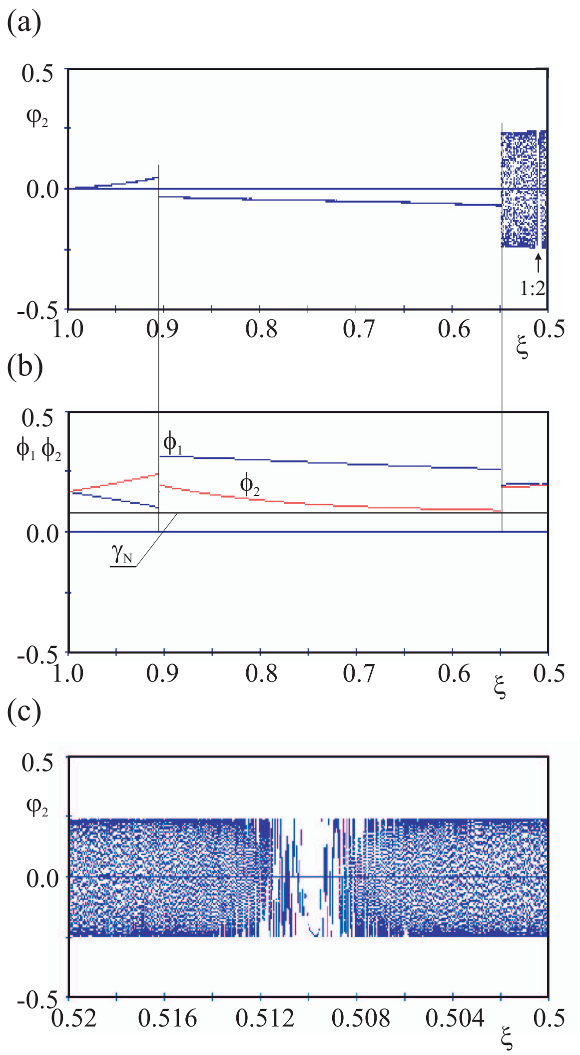

Fig. 5: (a) bifurcation diagram of the system (1) for decreasing ξ; $\varphi_{10} = \varphi_{20} = 0.26$, end with $\xi = 0.5$; (b) minima of amplitudes of pendulums oscillations; (c) enlargement of (a) in the neighborhood of 1:2 synchronization.

zero with the positive velocity is presented in Fig. 7(d). The closed curve at the Poincare map confirms that the behavior is quasiperiodic.

The subsequent change in the pendulums' configuration leads to 1:2 synchronization explained in Fig. 7(e) (the synchronization obtained by doubling the period of the pendulum 2 ($\xi = 0.51$)). Common period of pendulums oscillations is equal to the period of pendulum 1 T_1 and two periods of pendulum 2, i.e., $T = T_1 = 2T_2 = 0.94$. Noteworthy is the fact that in the case of nonmoving beam this synchronization would occur for $\xi = 0.5 = 1/2$ and $T_1 = 2T_2$. In the case of the oscillating beam this transition is shifted as due to the different pendulums' locations $\varphi_1 \neq \varphi_2$ and their different lengths $l_1 \neq l_2$ the resultant of the forces with which pendulums act on the beam is not always equal to zero.

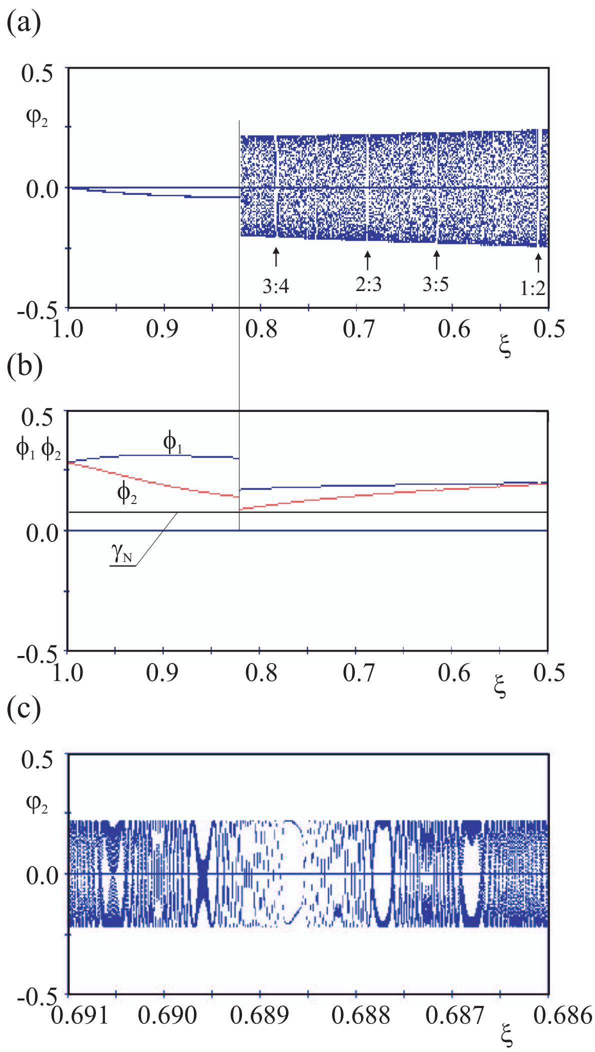

Fig. 6: (a) bifurcation diagram of the system (1) for increasing ξ; (b) minima of amplitudes of pendulums' oscillations; (c) enlargement of (a) in the neighborhood of 2:3 synchronization.

Fig. 5(b) explains why the different types of synchronization are visible along the bifurcation diagram of Fig. 5(a). The amplitudes of both pendulums Φ_1 and Φ_2 versus coefficient ξ are shown together with the value of the operations' angle of the escapement mechanism γ_N. Notice that for $\xi = 0.92$ when the in-phase 1:1 synchronization is replaced by the anti-phase 1:1 synchronization the amplitude of the pendulum 1 Φ_1 decreases to the value equal to γ_N. With the further decrese of ξ the escapement mechanism of pendulum 1 does not operate and the amplitude Φ_1 decreases due to the damping $c_{\varphi 1}$. The escapement mechanism starts to operate again and due to the large amplitude difference the system reach anti-phase 1:1 synchronization. With furher decrease of ξ the amplitude of the pendulum 2 Φ_2 decreses up to the critical value γ_N ($\xi = 0.55$) when the escapement mechanism of this pendulum is switched off. The perturbation which results from this switch off desynchronizes the pendulums. For smaller values of ξ, one observes the

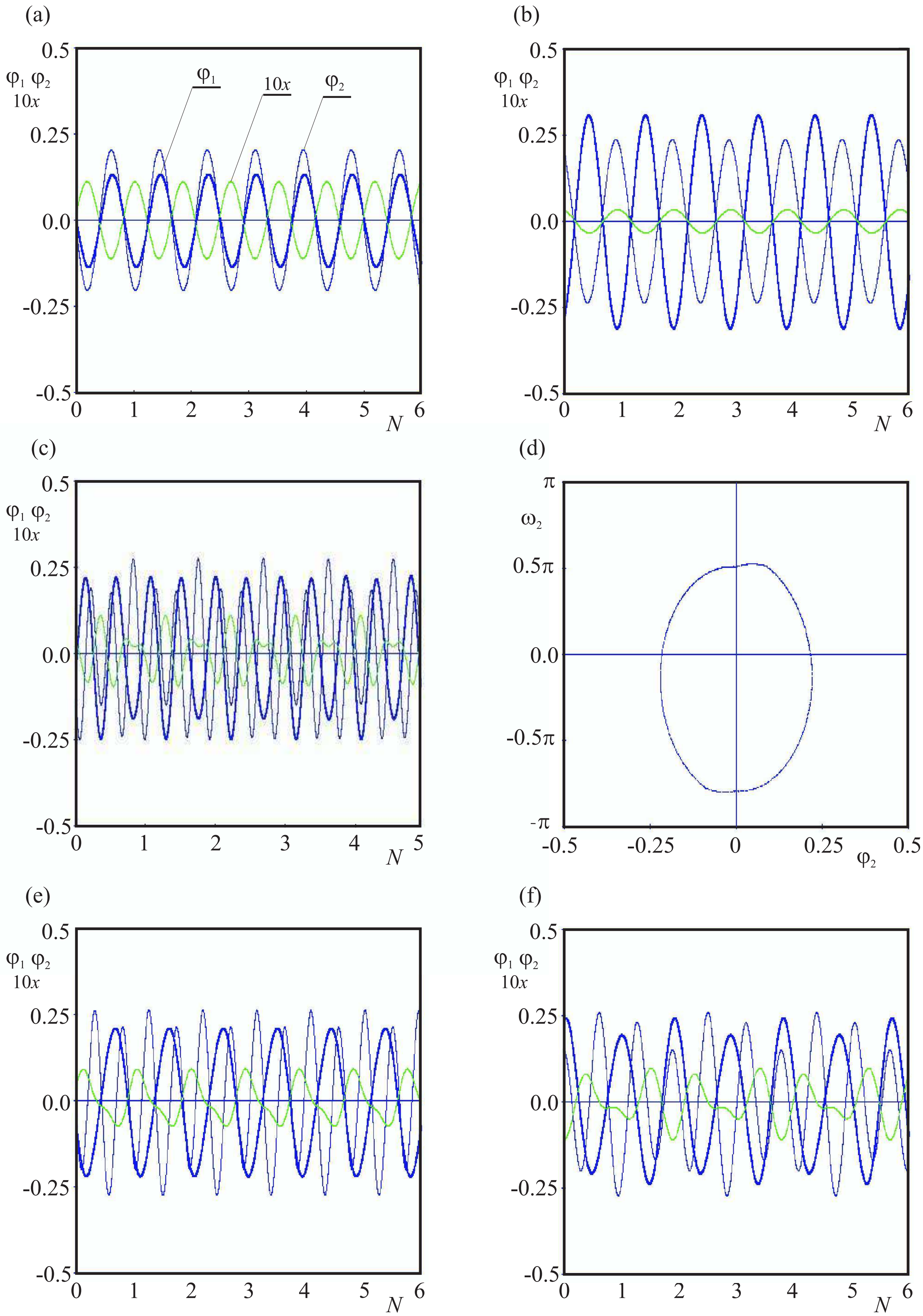

Fig. 7: Different types of synchronization of the clocks pendulums; (a) 1:1 in-phase synchronization for $\xi = 0.95$; (b) 1:1 anti-phase synchronization for $\xi = 0.95$; (c) time series of quasiperiodic motion for $\xi = 0.6871$; (d) Poincare map, (e) synchronization 1:2 for $\xi = 0.51$; (f) synchronization 2:3 for $\xi = 0.6887$.

quasiperiodic behavior. In this case the values of Φ_1 and Φ_2 are not constant for given ξ as can be seen in Fig. 7(c,d) so Fig. 5(c) shows their minimum values. As both minimun values of Φ_1 and Φ_2 are nearly twice larger than the value of γ_N the escapement mechanisms of both pendulums are not switched off during the quasiperiodic behavior. Fig. 5(c) shows the enlarged part of the diagram of Fig. 5(a). Synchronization 1:2 is visible on it as a single line in the interval $0.51 > \xi > 0.5095$. The length of this interval is equal to $\Delta\xi = 0.0005$, i.e., the given length of the pendulum 1, say $l_1 \approx 0.25[m]$ the length of pendulum 2 can change only by $0.0001[m]$, so the practical importance of this synchronization is rather low. In the last considered interval $0.5095 > \xi > 0.5$ the system (1) shows quasiperiodic behavior. With further decrease of ξ one can observe the 1:3 synchronization (not shown in Fig. 5(a)) for $\xi = 0.3375$. In this case the oscillation period common for both pendulums T is equal to the period of the pendulum 1 T_1 and three periods of pendulum 2, i.e., $T = T_1 = 3T_2$.

Fig. 6(a-c) shows the bifurcation diagram calculated for the increasing values of ξ (from 0.5 to 1.0). Differently to the case of decreasing ξ , the interval of quasiperiodic behavior is larger as can be seen in Fig. 6(a). Such a behavior exists for $0.5 < \xi < 0.82$ so in the interval $0.55 < \xi < 0.82$ one can observe two co-existing attractors; periodic (1:1 anti-phase synchronization) and quasiperiodic one. The quasiperiodic interval is characterized by the periodic windows, for example these visible for $\xi = 0.68875 (\approx 2/3)$, $\xi = 0.6168 (\approx 3/5)$, $\xi = 0.783 (\approx 3/4)$. Fig. 6(b) allows the determination of moments when the escapement mechanism is switched off. For the quasiperiodic behavior in the interval $0.5 < \xi < 0.82$ minimum values of Φ_1 and Φ_2 are larger than γ_N so the mechanism is not switched off. With the decrease of ξ the amplitude of the pendulum 2 decreases and is equal to γ_N for $\xi = 0.82$. For further increase of ξ the ecscapement mechanism of pendulum 2 is switched off, the quasiperiodic behavior is perturbed and the regime of anti-phase 1:1 synchronization is achieved. Fig. 6(c) shows the enlargement of the part of bifurcation diagram of Fig. 6(a). In the interval $0.6885 < \xi < 0.6888$ one observes 2:3 synchronization. This type of synchronization has been achieved by the doubling of the period of pendulum 1 T_1 and tripling the period of pendulum 2 T_2. The time series characteristic for this behavior are shown in Fig. 7(f) ($\xi = 0.6887$). The period of oscillations common for both pendulums is equal to $T = 2T_1 = 3T_2 = 1.91$. For $\xi = 0.783$ we observe 3:4 synchronization and for $\xi = 0.6158$ 3:5 synchronization. All these synchronized regimes exist in the very narrow intervals of ξ.

The bifurcation diagrams of Figs 5(a-c) and 6(a-c) show the existence of two different attractors for some ξ-intervals. For $1.0 > \xi > 0.92$ one can observe 1:1 in-phase and 1:1 anti-phase synchronization, and for $0.82 > \xi > 0.55$ 1:1 anti-phase synchronization and quasiperiodic behavior (or 1:2, 2:3, 3:4, ... synchronization in the narrow windows).

The question how the co-existing attractors are sensitive to the external perturbations cannot be generally answered as both phase and parameter spaces of the system are high-dimensional. We partially address this problem by performing the following experiment. Having assumed that the system is operating on one of the coexisting attractor we perturb the state of pendulum 2 and observe to which attractor the system will approach. Such a procedure allows the estimation of the basins of attraction of the coexisting attractors shown in Fig. 8(a-c).

Perturbation of the state of pendulum 2 means the change of its position to the new state given by φ_{20} and $\dot{\varphi}_{20}$. Other system parameters, i.e., φ_1, $\dot{\varphi}_2$, x and $\dot{x}$ are not changed at the moment of perturbation. Then the system (1) performs the transient evolution which leads to one of the coexisting attractors. Notice that such perturbation can influence the acting of the escapement mechanism. If in the moment of perturbation the mechanism is in the first stage (see Eqs. in Sec. 3.1) and new value of φ_{20} is larger than γ_N, the mechanism moves to step 2. When in the unperturbed state the mechanism is in step 2 and new value of φ_{20} is smaller than $-\gamma_N$, the mechanism goes to step 1. For other cases the perturbation has no influence on the acting of the escapement mechanism.

In the example presented in Fig. 8(a) we assume that the system (1) performs 1:1 in-phase synchronization for $\xi = 0.95$ (the coexisting attractor is anti-phase 1:1 synchronization). When the system has been in the state given by: $\varphi_1 = 0.0$, $\dot{\varphi}_1 = 0.9616$, $\varphi_2 = 0.01808$, $\dot{\varphi}_2 = 1.4945$, $x = -0.001635$, $\dot{x} = -0.080947$ pendulum 2 has been perturbed to the state given by (φ_{20}, $\dot{\varphi}_{20}$). The initial perturbations (φ_{20}, $\dot{\varphi}_{20}$) after which system (1) returns to the 1:1 in-phase synchronization are shown in black and these after the system goes to anti-phase 1:1 synchronization are shown in black.

Fig. 8(b) presents the results obtained for $\xi = 0.95$ and the initial state given by $\varphi_1 = 0.0$, $\dot{\varphi}_1 = 1.9525$, $\varphi_2 = -0.0207$, $\dot{\varphi}_{20} = -1.48$, $x = -0.000267$, $\dot{x}_0 = -0.0221$, i.e., the system has been on the 1:1 anti-phase synchronization attractor at the moment of perturbation. The basins of in-phase and anti-phase synchronization are shown respectively in white and black.

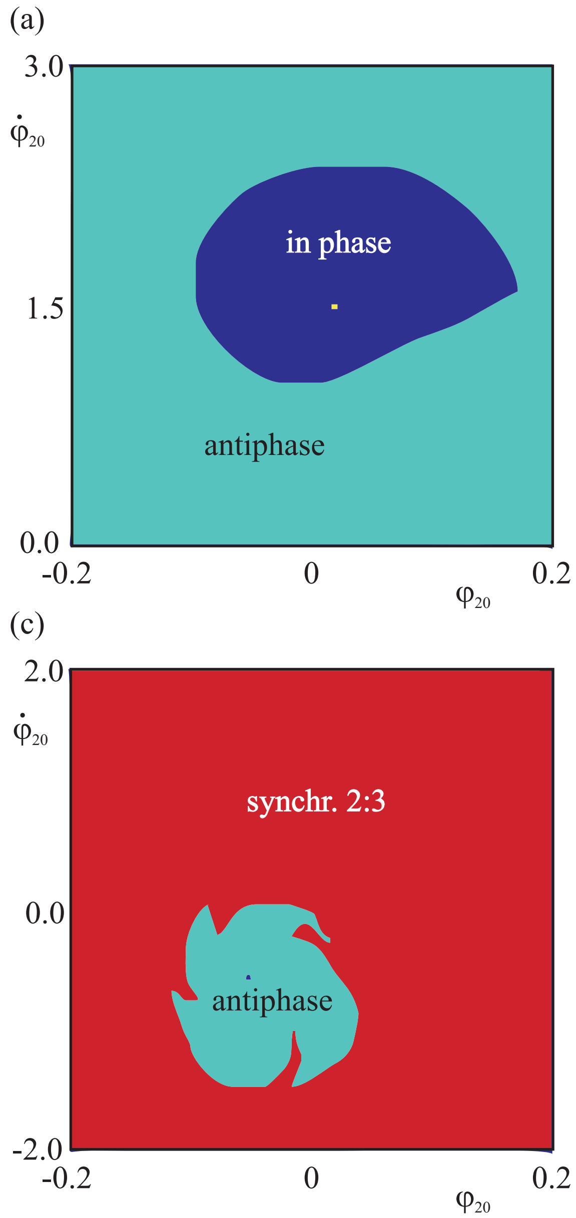

Fig. 8: Basins of attraction of the co-existing attractors; (a) $\xi = 0.95$; (b) $\xi = 0.95$; (c) $\xi = 0.6887$.

For $\xi = 0.6887$ system (1) has two attractors; 1:1 anti-phase and 2:3 synchronization (see Fig. 8(c). In the time of perturbation the system has been in the state: $\varphi_1 = 0.0$, $\dot{\varphi}_1 = 1.8385$, $\varphi_2 = -0.05257$, $\dot{\varphi}_2 = -0.558$, $x = 0.00004$, $\dot{x}_0 = -0.0559$ exhibiting 1:1 anti-phase synchronization. The basins of 1:1 anti-phase and 2:3 synchronization are shown respectively in white and black in Fig. 8(c).

4. Conclusions

We show that two clocks hanging on the same beam can synchronize both in-phase and anti-phase. The synchronization has been observed when the beam has been located on the rolls which allowed its horizontal movement. Additionally, the clocks have to be hanged in such a way to avoid (reduce) the spherical motion of their cases.

Although the clocks in experiment have been selected in such a way to be as identical as possible, they have some small differences in the pendulum lengths and periods (too small to be identified in the Huygens' times). We perform computer simulations to answer the question: how the nonidentity of the clocks influences the synchronization process. It has been shown that even the clocks with significantly different periods of oscillations can synchronize, but their periods are modified by the beam motion so they are obviously no more accurate. Our numerical studies show that the anti-phase synchronization is dominant in the parameters space.

Additionally, to in-phase and anti-phase synchronization we identify the regions of quasiperiodic oscillations of pendulums and phase locking regions of 1:2, 2:3, 3:4, ... synchronization. It has been shown that when the amplitude of oscillations of one of the pendulums decreases (as the result of the beam motion) to the value smaller than N the escapement mechanism is temporarily switched off and the systems move to the coexisting attractor after this mechanism is switched on again.

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