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Experimental verification of a hybrid dynamical model of the church bell

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ABSTRACT

In this paper we present a hybrid model of church bell. The dynamical system of yoke-bell-clapper is nonlinear and discontinuous. We use the Lagrange equations of the second type to derive formulas that describe the system's motion. The energy between the bell and the clapper is transmitted via impacts, here modeled using a coefficient of energy restitution. The values of the system's parameters have been determined basing on the measurements of the biggest bell *"The Heart of Lodz"* in the Cathedral Basilica of St Stanislaus Kostka, Lodz, Poland. Using the same bell we also validate the model by comparing the results of numerical simulations with experimental data. The presented results show that the described model is a reliable predictive tool which can be used both to simulate the behavior of the existing yoke-bell-clapper systems as well as to design the yokes and predict the motion of new bells.

1. Introduction

The history of bells as musical instruments started more than 5000 years ago in China and since then they have spread around the whole world. Bells usually play an important cultural role and till today their sounds announce the major events. Throughout centuries bell-founders and craftsmen developed their knowledge about bells basing on experience, intuition and tradition. Although the design of a bell, a clapper and a belfry has been improved for ages, their mathematical modeling has been encountered only recently. Bells are mounted in a number of ways that are characteristic for local tradition. In Europe one can distinguish three different types of swinging bells: Central European, English and Spanish. In the first of them, the bells tilt on their axis and maximum amplitude of oscillation is usually below 90°. In the English system the bells perform nearly a complete rotation (the bell stops close to the upper position). Conversely, in the Spanish system the bells rotate continuously in the same direction.

The first attempt to describe the bell's behavior using the equation of motion was made by Veltmann in 19th century [1,2]. His work was stimulated by the failure of the famous Emperor's bell in the Cologne Cathedral when the clapper remained always on the middle axis of the bell instead of striking it. Veltmann explained the reason of this phenomenon using a simple model developed basing on the equations of a double physical pendulum. Although he proved that even simple mathematical modeling can help bell-founders and engineers who build bell supports and belfries, his work has not been continued for many years. About a century later Heyman and Threlfrall [3] use a similar double pendulum model to estimate inertia forces induced by a swinging bell. Muller [4] and Steiner [5] analyze the dynamic interactions between bells and bell towers and describe the dynamic forces appearing in the bells mounted in the Central European manner. Their work is continued by Schutz [6] and by the authors of the German DIN standard [7] who develop a semi-empirical description of the forces acting on the bells' supports. The knowledge of the loads induced by the ringing bells is crucial during the design and restoration processes of the belfries. There are also similar studies concerning the English system [8,9] as well as the Spanish system [10,11].

Ivorra et al. [12,13] present the influence of the mounting layout on the dynamic reactions' forces produced by the swinging bells. They perform numerical simulations and laboratory that show that the forces transmitted to the supporting structure are significantly lower in the Spanish system than in the English and Central

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European ones. The studies described in Refs. [14,15] prove that quite often a minor modification in the support's design can significantly decrease the probability of the damage of the bell's tower.

The dynamic system of a yoke-bell-clapper and a supporting structure can be considered on a multiple scales. The publications mentioned above describe the dynamic forces acting on bell supports neglecting the analysis of the clapper to the bell impacts as they do not influence bell towers significantly. One can find a number of publications focusing on the ringing process itself which provide an important source of information and help to develop the knowledge about the impacts of the rigid bodies. In 2010 Klemenc et al. [16] study contact zone during the clapper-to-bell impact. They introduce a simple model consisting of a steel cylinder and a bronze block and perform the tests to assess under the laboratory-controlled conditions the consequences of the repetitive hits. Next, they compare the results obtained from the finite-element method with the experimental data and prove that numerical simulations of the process give satisfying and comparable results. The authors continue their research and contribute with the analysis of the full-scale finite-element model of the clapper to the bell impacts [17].

In Ref. [18] Meneghetti and Rossi present the lumped parameter model of the church bells mounted in the Central European system. The authors compare two different modeling approaches: the first assumes the discontinuity in the equations of motion and the second maintains the integration of the equations during the whole contact time. The authors validate the proposed models by comparing the results of numerical simulations with the experimental data in terms of impact acceleration and bell period. Woodhouse and McClenahan [19] analyze the dynamics of church bells working in the English manner. The authors focus on the phase of motion involving free oscillations of the bell and the clapper. Basing on the series of numerical simulations the authors develop the charts that show whether a given bell can ring “right”, “wrong”, or both depending on the values of two introduced dimensionless parameters. Highlighting the practical significance of the presented results the authors also indicate the need to investigate what factors are important beyond the values of two introduced parameters.

In this paper we propose a hybrid dynamical model of a yoke-bell-clapper system which is able to simulate the dynamic behavior of church bells: both in free motion of the bell and the clapper as well as in normal working conditions when recurrent impacts occur. The model incorporates Lagrange equations of motion and a simple discreet model of the impact based on the coefficient of restitution. To determine the values of the parameters of the model we perform the measurements of the biggest bell of the Cathedral Basilica of St Stanislaus Kostka – *“The Heart of Lodz”*. In order to increase the accuracy and reliability of the simulations we expand the classical model by taking into consideration the bell's linear motor propulsion and the energy dissipation due to the torsional damping in both the bell's and the clapper's pin joints.

After derivation and fine-tuning, we test and validate the model by comparing the results of numerical simulations with the experimental data collected during typical working conditions of *“The Heart of Lodz”*. We compare the trajectories of the bell and the clapper instead of focusing only on the period of the bell or periods between the successive impacts. Numerical results prove that the presented model is a reliable predictive tool which does not require high computational power or long calculation times.

The paper is organized as follows: in Section 2 we present the geometrical measures which are used to describe the yoke-bell-clapper system, in Section 3 we derive the continuous model to simulate free-motion conditions and validate the energy dissipation model. Section 4 contains the description and validation of the discreet model of the clapper to the bell impact. Finally, in Section 5 we test the full hybrid model by comparing the results of numerical simulations with the experiment and summarize our work in Section 6.

2. Geometry of the yoke-bell-clapper system

The model that is presented in this paper is build up based on the analogy between the freely swinging bell and the motion of the equivalent double physical pendulum. The first pendulum has fixed axis of rotation and models the yoke together with the bell that is mounted on it. The second pendulum is attached to the first one and imitates the clapper. The proposed model involves eight physical parameters. Unfortunately, the precise values of the parameters need to be determined specifically for the purpose. It is because the bell-founders usually use wooden or steel templates that usefullness has been proven over the centuries and most of them do not create any technical documentation for the bells they cast. Therefore, to get realistic values of the existing system's parameters we have performed detailed measurement of the bell in the Cathedral Basilica of St Stanislaus Kostka.

The photo of the bell that we consider is presented in Fig. 1(a). In Fig. 1(b and c) we show the schematic model of the bell indicating the position of rotation axes of the bell o_1, the clapper o_2 and present the parameters involved in the model. For simplicity, during the development of the model and henceforth, we use the term “bell” with respect to the combination of the bell with its yoke which we treat as one solid element.

Parameter L describes the distance between the rotation axis of the bell and its center of gravity (point C_b), l is the distance between the rotation axis of the clapper and its center of gravity (point C_c). The distance between the bell's and the clapper's axes of rotation is given by parameter l_c. The mass of the bell is described by parameter M, while parameter B_b characterizes the bell's moment of inertia referred to its axis of rotation. Similarly, parameter m describes the mass of the clapper and B_c stands for the clapper's moment of inertia referred to its axis of rotation.

The considered model has two degrees of freedom. In Fig. 1(d) we present two generalized coordinates that we use to describe the state of the system: the angle between the bell's axis and the downward vertical is given by Ref. φ_1 and the angle between the clapper's axis and the downward vertical by Ref. φ_2. Parameter α (see Fig. 1(d)) is used to describe the clapper to the bell impact condition which is as follows:

$$|\varphi_1 - \varphi_2| = \alpha \quad (2.1)$$

Synonymously, the collision between the bell and the clapper occurs when the difference between the bell's and the clapper's angular displacements is equal to α.

As aforementioned, the values of all parameters described above have been measured by us specifically to receive reliable dynamical model. Thanks to the kindness of Mr Zbigniew L. Felczyński – the bell-founder who created *“The Heart of Lodz”*, we have been able to perform the detailed measurements of the bell's template and estimate the density of bronze alloy from which it has been cast. Basing on the collected data we have created three dimensional model of the bell which we have used to determine the mass of the bell, its moment of inertia and the location of the center of mass. We have also performed detailed measurements of the installed bell to determine the geometry of the yoke and the mounting layout of the bell. We have complemented the gathered data with the data received from Mr. Paweł Szydlak (the designer of the

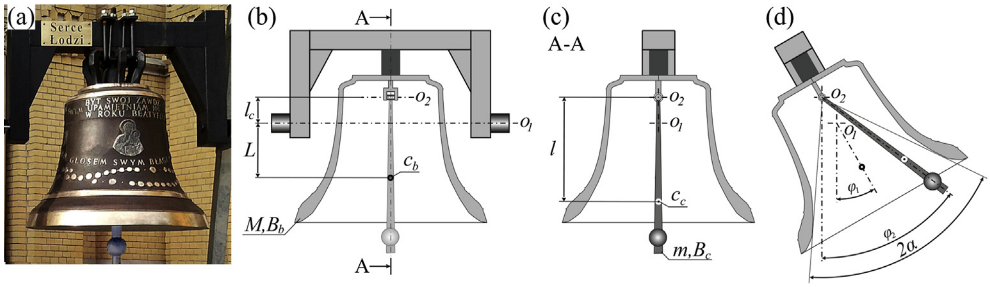

Fig. 1. *"The Heart of Lodz"* the biggest bell in the Cathedal Basilica of St Stanislaus Kostka (a) and its schematic model (b,c,d) along with physical and geometrical quantities involved in the mathematical model of the system.

clapper, the bell's yoke and the propulsion mechanism) and developed the full three dimensional model of the yoke-bell-clapper system. Using the above methods we have determined the following values of system's parameters: $M = 2633$ [kg] (mass of the bell equals 2376 [kg] and the yoke weights 257 [kg]), $m = 57.4$ [kg], $B_b = 1375$ [kgm^2], $B_c = 45.15$ [kgm^2], $L = 0.236$ [m], $l = 0.739$ [m], $l_c = -0.1$ [m] and $\alpha = 30.65^\circ = 0.5349$ [rad]. It is important to point out that the parameters values presented above are encumbered with errors due to the casting inaccuracy, small bell's ornaments which are not included in the model and residual unbalance which is always present in real systems and etc. Nevertheless, taking into account that bell-founding is still more craft art than engineering design we can assume that the accuracy of our measurements cannot be increased significantly. Hence, one of our goals is to verify if we are able to accurately predict the behavior of the existing bell even though its parameters are determined with limited precision.

3. Equations of motion − modeling of free motion of the system and the model of the bell's propulsion

In this section we describe the mathematical model that we use to simulate the oscillatory motion of the investigated yoke-bell-clapper system. We employ Lagrange equations of the second type to derive the equations of motion of the system described in the previous section. The total kinetic energy T and potential energy V of the considered system are given by the following formulas:

$$T = \frac{1}{2}\left(B_b + ml_c^2\right)\dot{\varphi}_1^2 + \frac{1}{2}B_c\dot{\varphi}_2^2 + ml_cl\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_2 - \varphi_1), \tag{3.1}$$

$$V = (ML + ml_c)g(1 - \cos\varphi_1) + mlg(1 - \cos\varphi_2), \tag{3.2}$$

where g stands for gravity.

Church bells have various types of propulsion mechanisms. Traditionally bells are driven and controlled by ringers who adjust the exerted force basing on their experience and intuition. Nowadays, most of the bells are driven by an electric motors. In early realizations of electric propulsion the yoke is connected with the sprocket and driving torque is transmitted from an AC motor to the bell by a chain transmission. Recently, there is a tendency to use linear motors as they imitate manual control better and make the phase of free motion of the bell possible. Our model is based on the bell which is propelled by a modern linear motor. The motor is active and excites the bell when its deflection from vertical position is smaller than 12°. The torque generated by the motor depends on the degree of coverage between the "unrolled" stator and rotor. We introduce the linear motor excitation to the mathematical model by the generalized momentum $M_t(\varphi_1)$ given by the following piecewise formula:

$$M_t(\varphi_1) = \begin{cases} T\,\mathrm{sgn}(\dot{\varphi}_1)\cos(7.5\varphi_1), & \text{if} \quad |\varphi_1| \le \dfrac{\pi}{15} \\ 0, & \text{if} \quad |\varphi_1| > \dfrac{\pi}{15} \end{cases} \tag{3.3}$$

Although, the above expression is not an accurate description of the excitation momentum generated by the linear motor, it is able to reproduce the end effect of the linear motor (using the term $\cos(7.5\varphi_1)$) and the propulsion control system which is responsible for reversing the direction of the driving torque (by term $\mathrm{sgn}(\dot{\varphi}_1)$). Therefore, it captures the general characteristic of the bell's propulsion.

In the considered model we describe the energy dissipation using Rayleigh dissipation function D which accounts for the torsional damping in both the bell's and the clapper's pin joints.

$$D = \frac{1}{2}D_b\dot{\varphi}_1^2 + \frac{1}{2}D_c(\dot{\varphi}_2 - \dot{\varphi}_1)^2. \tag{3.4}$$

The damping in a pin joint is composed of the viscous and dry friction damping [20]; we neglect the dry friction component (see next Subsection for the experimental validation). Thanks to that, we receive continuous model of damping which, as proved in Subsection 3.1, is capable of imitating the energy dissipation in the pin joints of the bell and the clapper.

Using formulas (3.1)–(3.4) we derive two coupled second order ODEs that describe the motion of the considered yoke-bell-clapper system:

$$\begin{aligned}&\left(B_b + ml_c^2\right)\ddot{\varphi}_1 + ml_cl\ddot{\varphi}_2\cos(\varphi_2 - \varphi_1) - ml_cl\dot{\varphi}_2^2\sin(\varphi_2 - \varphi_1)\\ &+ (ML + ml_c)g\sin\varphi_1 + D_b\dot{\varphi}_1 - D_c(\dot{\varphi}_2 - \dot{\varphi}_1) = M_t,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&B_c\ddot{\varphi}_2 + ml_cl\ddot{\varphi}_1\cos(\varphi_2 - \varphi_1) + ml_cl\dot{\varphi}_1^2\sin(\varphi_2 - \varphi_1) + mgl\sin\varphi_2\\ &\quad + D_c(\dot{\varphi}_2 - \dot{\varphi}_1) = 0.\end{aligned} \tag{3.6}$$

There are eleven parameters involved in the mathematical model presented above. The values of eight parameters have been determined by the measurements of the bell at rest, namely: M, m, B_b, B_c, L, l, l_c and α. To determine the damping coefficients in pin joints of the bell D_b and the clapper D_c we perform a series of experiments described in next subsection. The value of parameter T which describes the maximum force generated by the driving linear motor is estimated basing on the data collected during normal working conditions of the bell. For integration of the model described above we use the fourth-order Runge−Kutta method. ODEs 6 and 7 together with the discreet model of the impact described in the next section create a hybrid dynamical system able to simulate the behavior of the church bell.

3.1. Determination of the viscous damping coefficients in the pin joints

As aforementioned, in the model presented in this paper we assume the viscous torsional damping in the bell's and the clapper's pin joints and neglect dry friction resistance. To validate the assumed model of energy dissipation and determine viscous damping coefficients we perform a series of experiments on the bell installed in the Cathedral Basilica of St Stanislaus Kostka. Using a high-speed camera (Basler piA640-210gm) which enables to save 200 frames per second we record small amplitude free oscillations of the clapper. We use black-and-white stickers to mark the central axis of the clapper and indicate the reference length. In Fig. 2(a) we present a snapshot from one of the recordings and show the position of the assumed coordinates system origin, the clapper's marker and the reference lengths.

Using free-source Kinovea software [21] track the marker's location in time. Basing on the recorded movies we prepare the data-sheets with the marker's abscissa and ordinate depending on time. To create the time traces of the clapper's angular displacement we use Mathematica software and assume that the bell is at rest. Hence, the position of the clapper's axis of rotation is constant. In order to assess the accuracy of the received data we calculate the standard deviation of the distance between the clapper's marker and the axis of rotation. The standard deviation for all collected data is equal to 0.0006 [m]. Therefore, we assume that our measurements are accurate and the collected data are reliable.

The logarithmic decrement method is used to estimate the value of the viscous damping coefficient in the clapper's pin join for free vibration traces. We analyze separately each set of data collected during one recording and then calculate the average of received values. We assume the referred average value $D_c = 4.539$ [Nms] as a damping coefficient of the clapper's pin join. In Fig. 3 we present the comparison between the two sets of data collected during the experiments and the results of numerical simulations of the system with the determined value of D_c parameter. Based on the results presented in Fig. 3 one can say that the proposed model ensures the accuracy of the simulations and there is no need to expand the model with dry friction.

To determine the viscous damping coefficient in the bell's pivot point given by parameter D_b we use the procedure similar to the one described above. We examine small amplitude free oscillations of the bell using the high-speed camera. In Fig. 2(b) we present a snapshot from one of the recordings and indicate the applied position markers. Basing on the mentioned recordings and using the tools described above we obtain the time traces of the bell. The standard deviation of the distance between the bell's pin joint and its position marker is equal to 0.0011 [m] for all collected data. Hence, the obtained time traces can be regarded as reliable.

During the motion of the bell the energy is dissipated in both the bell's and the clapper's pin joints. When the bell is examined the clapper barely moves so, for simplicity, we assume that the clapper's angular velocity is constant and equal to 0 ($\dot{\varphi}_2 = 0$). Therefore, the average value of damping coefficient determined using the logarithmic decrement method is the sum of coefficients in both pin joints. After deducting the earlier determined value of parameter D_c we get the value of damping coefficient in the bell's pin joint $D_b = 26.68$ [Nms]. In Fig. 4 we compare the experimentally obtained time traces of the bell with the results of numerical simulations. Similarly as for the clapper, one can see good agreement of the presented results which proves the reliability of the proposed model.

Interestingly, the value of the bell's pin joint damping coefficient may seem surprisingly small in comparison with the damping in the clapper's pivot point. It is due to the fact that the yoke is equipped with high class bearings to protect the linear motor because even small deviation of the bell's axis of rotation position can cause its damage.

4. Modeling of the clapper to the bell impact

Having considered free motion conditions; in this section we describe the model of the clapper to the bell impact. One can distinguish many different approaches coping with the collisions between the rigid bodies. There is a number of continuous models such as Kelvin-Voigt viscoelastic model [22,23], Hertz contact model [24,25] or Mindlin Deresiewicz model [26,27]. These models are based on differential equations which describe the course of collisions. Solving the equations we are able to obtain time histories of different dynamic quantities during the impact. The biggest disadvantage of these methods is longer computational time and more complex calibration. However we may expect to obtain better accuracy in return for extra effort. discrete impact models such as Newton [28,29], Poisson [30], or Brach [31,32] are based on the definition of a coefficient of restitution. These models consist of an algebraic formulas which enable to calculate the results of collision basing on the description of pre-collision dynamical state of the system. discrete methods are often the first choice as they are straightforward in application and in most cases enable to simulate correctly the global dynamic behavior of the impacting systems.

Meneghetti and Rossi [18] compare the continuous and discrete models of the clapper to the bell impact. They prove that simple discrete approach based on the Newtonian coefficient of restitution gives results comparably accurate to the ones obtained using the theory of Hertzian contact. Therefore, in this paper we employ the discrete impact model that we describe below.

The collision condition which is defined by the geometry of the system is as follows:

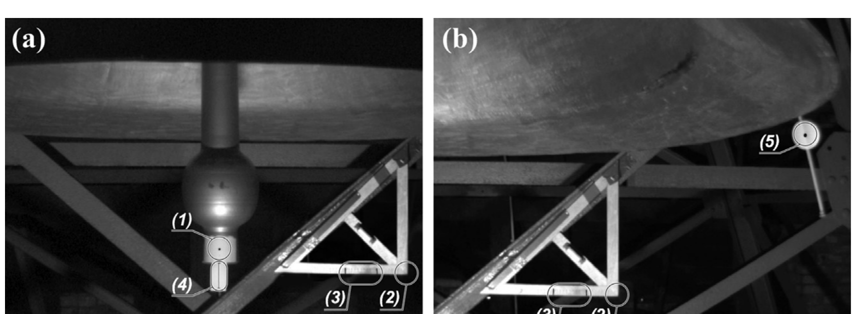

Fig. 2. Snapshots form high-speed camera recordings used to determine viscous damping coefficients in the clapper's (a) and the bell's (b) pin joints. Abbreviation *(1)* refers to the marker used to determine the position of the clapper, *(5)* points at the bell's position marker, *(2)* is the origin of coordinates system. Indications of reference lengths are pointed by *(3)* and *(4)*.

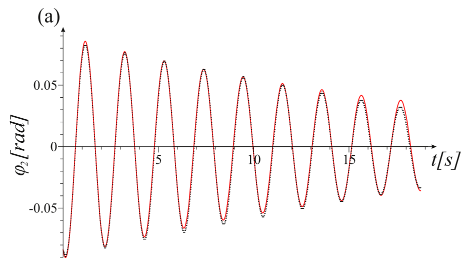

Fig. 3. Time traces of the clapper's small amplitude free oscillations. Black dots correspond to the data obtained experimentally while red lines present the results of numerical simulations. Initial conditions for subplot (a) are as follows: $\varphi_1 = 0$, $\dot{\varphi}_1 = 0$, $\varphi_2 = -0.0810$, $\dot{\varphi}_2 = -0.0997$, and for subplot (b): $\varphi_1 = 0$, $\dot{\varphi}_1 = 0$, $\varphi_2 = -0.0655$, $\dot{\varphi}_2 = 0.0260$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$|\varphi_1 - \varphi_2| = \alpha \tag{4.1}$$

When condition (4.1) is fulfilled we stop the integration process. Then, instead of analyzing the collision course, we restart the simulation updating the initial conditions of equations (3.5) and (3.6) by switching the bell's and the clapper's angular velocities from the values before the impact to the ones after the impact. The angular velocities after the impact are obtained taking into account the energy dissipation and the conservation of the system's angular momentum. In the considered model we assume that the clapper's relative kinetic energy after the impact is the known part of the value before the impact, which can be written as follows:

$$\frac{1}{2}B_c\left(\dot{\varphi}_{2,AI} - \dot{\varphi}_{1,AI}\right)^2 = k\frac{1}{2}B_c\left(\dot{\varphi}_{2,BI} - \dot{\varphi}_{1,BI}\right)^2, \tag{4.2}$$

where index *AI* stands for "after impact", index *BI* for "before impact" and parameter k is the coefficient of energy restitution. In our simulations we assume $k = 0.05$ referring to the series of experiments performed by Rupp et al. [33]. specifically to evaluate the value of the restitution coefficient for the clapper to the bell impacts. The value of parameter k is relatively small for the collision of the rigid bodies. It is due to the fact that when the clapper hits the bell the significant amount of its kinetic energy is transferred into the bell's vibrations and into acoustic energy. In fact the bell, as a musical instrument, has the shape which easily excites the vibrations of a certain desired frequency.

In normal working conditions of *"The Heart of Lodz"* impacts occur when the driving motor is not active. Hence, there is no external force acting on the system. Therefore, during the impacts the angular momentum of the system is conserved and we are able to write the second equation that is used to determine angular velocities after the impact. The following equation states the system's angular momentum conservation during the impact and describes the relation between the bell's and the clapper's velocities just before and just after the impact:

$$\begin{aligned}&\left[B_b + ml_c^2 + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{1,BI}\\ &\quad + \left[B_c + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{2,BI}\\ &= \left[B_b + ml_c^2 + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{1,AI}\\ &\quad + \left[B_c + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{2,AI}\end{aligned} \tag{4.3}$$

When impact condition (4.1) is fulfilled we mark down the bell's and the clapper's angular velocities before the impact and using equations (4.2) and (4.3) calculate the values of angular velocities after the impact. Then, we use the obtained velocities values as initial conditions and restart the integration.

4.1. Verification of the impact model

In this subsection we present the experimental verification of the clapper to the bell impact model described above. During the experiment we want to replicate the idealized ringing conditions. The bell and the clapper are set in motion manually. Thanks to that we are able to preserve angularity of the collision typical for *"The Heart of Lodz"* mounting layout and avoid any preliminary strikes. We want to avoid any prior contact as it will cause vibrations of the bell and its supporting structure. It is obvious that such phenomena are present during the normal operation of the bell but they are not included in the impact model which we want to validate by the experiment. The influence of these phenomena on the bell's and the clapper's dynamics is presented and analyzed in Section 5 where we investigate the full model of the system in normal working conditions.

Using high-speed camera we record 200 frames per second recording of the clapper to the bell impact. Then, using the methods described in Subsection 3.1, we obtain time traces of the bell and the clapper. Next, we read off the deflection and the angular velocity of the bell and the clapper 0.3 s before the collision and use them as the initial conditions for the numerical model. In Fig. 5 we

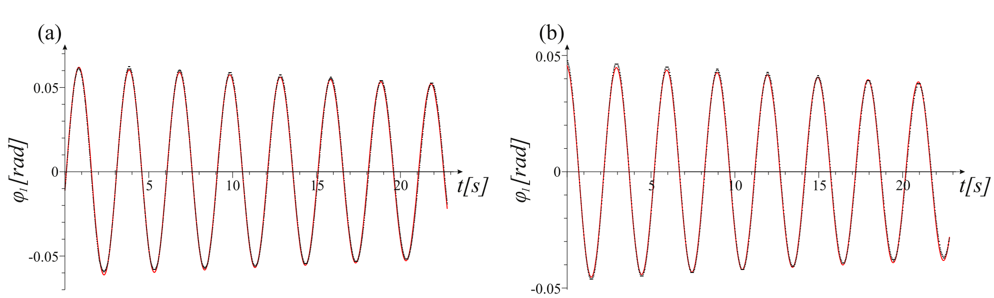

Fig. 4. Time traces of the bell's small amplitude free oscillations. Black dots correspond to the data obtained experimentally while red lines present the results of numerical simulations. Initial conditions for subplot (a) are as follows: $\varphi_1 = -0.0109$, $\dot{\varphi}_1 = 0.1288$, $\varphi_2 = 0$, $\dot{\varphi}_2 = 0$, and for subplot (b): $\varphi_1 = 0.0456$, $\dot{\varphi}_1 = -0.0121$, $\varphi_2 = 0$, $\dot{\varphi}_2 = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

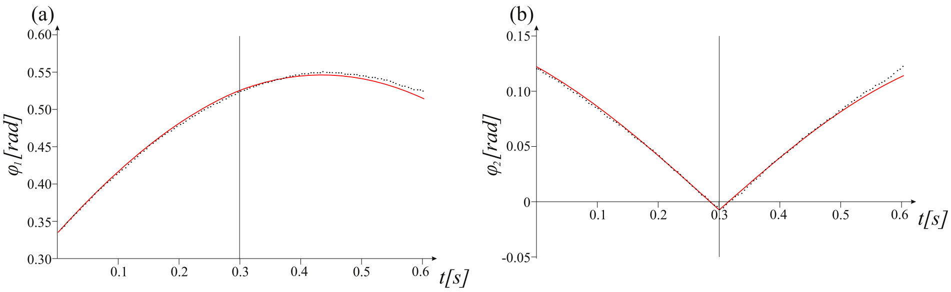

Fig. 5. Time traces of the bell (a) and the clapper (b) showing their behavior prior and after the collision. Black dots correspond to the data obtained experimentally while red lines present the results of numerical simulation. Initial conditions for the numerical integration are as follows: $\varphi_1 = 0.3373$, $\dot{\varphi}_1 = 0.9013$, $\varphi_2 = 0.1219$, $\dot{\varphi}_2 = -0.3096$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

present the comparison between the result of numerical simulation (depicted by the red lines) and the data collected experimentally (black dots). Analyzing Fig. 5 one can say that the discrete model of the impact is fully capable of simulating the clapper to the bell impact and that the assumed value of the coefficient of energy restitution $k = 0.05$ is correct. Moreover, simulations prove that the response of the system barely changes for small alterations of parameter k (±20%). Hence, we claim that there is no need to further adjust the value of k for the considered bell.

5. Numerical simulations of the system in normal ringing conditions

Equations (3.5) and (3.6) combined with the discrete impact model presented in Section 4 provide complete description of the yoke-bell-clapper system's behavior. So far, both continuous and discrete part of the proposed hybrid model have been verified experimentally. In this section we investigate the full model focusing on normal ringing conditions. Our study is based on the bell in the Cathedral Basilica of St Stanislaus Kostka and we check the reliability of the model by comparing the results of numerical simulations with experimentally obtained time traces of *"The Heart of Lodz"*.

We launch the linear motor propulsion and after a start-up procedure, when the amplitude of the bell's motion stabilizes, we begin the recording with the high-speed camera (Basler piA640-210gm). We use black-and-white stickers to mark the position of the bell, the clapper and indicate the reference length. We are not able to track the bell's position incessantly, because for $\varphi_1 \varepsilon <-0.15, 0.30>$ the bell's marker is obscured by the supporting structure. Nevertheless, the collected data is sufficient to illustrate the bell's behavior.

We use Kinovea software to create – basing on the recordings – the data-sheets with markers' abscissa and ordinate depending on time. Then, we process the data in Mathematica software. The position of the bell's pivot point is constant and we can easily calculate the bell's deflection from the downward vertical position. Using time trace of the bell and knowing the distance between the bell and the clapper swing axes we determine how the location of the clapper's pin joint changes in time. Afterward, using time dependent coordinates of the clapper's marker we are able to obtain its time trace. To evaluate the accuracy of the received time traces we compute the standard deviation of the distances between the markers and the axes of rotation for both the bell and the clapper. The standard deviation for the bell is equal to 0.0017 [m] and for the clapper to 0.0082 [m]. Hence, we claim that the accuracy of the obtained time traces is satisfactory.

As aforementioned in Section 3 we use the time traces of the bell to determine the value of parameter T which describes the maximum torque generated by the linear motor. First, we estimate the amplitude of the bell's oscillations by calculating the absolute values of the bell's maximum and minimum deflections recorded in each period of oscillations. Then, by the series of numerical simulations we adjust the value of parameter T to achieve the amplitude of the bell's motion equal to the experimentally determined value with an accuracy of 4 digits. This gives us the value of $T = 229.6$ [Nm]. Whereupon we have the values of all parameters involved in the numerical model of *"The Heart of Lodz"*.

We perform the numerical simulations using our own C++ code based on the fourth-order Runge–Kutta method. We start the integration with the fixed step $(\frac{1}{12\times 360})$ and monitor the value of the difference between the angular displacement of the bell and the clapper to be aware when the clapper is close to hit the bell. When the following condition is fulfilled:

$$\alpha - |\varphi_1 - \varphi_2| < 0.01, \tag{5.1}$$

we decrease the step of integration 4 times $(\frac{1}{48\times 360})$ and continue the integration up to the moment when:

$$\alpha - |\varphi_1 - \varphi_2| < 0.001, \tag{5.2}$$

If condition (5.2) is fulfilled we assume the collision occurre. We stop numerical integration, read off the angular velocities of the bell and the clapper and use the discreet model described in Section 4 to calculate the velocities after impact. Then, we use the obtained velocities as initial conditions and restarted the integration of the differential equations. Notice that the impact model swaps only the angular velocities and does not affect the angular position of the bell and the clapper. Therefore, in the first few steps of the integration impact condition 12 can still be fulfilled as it is defined with finite precision.

In Fig. 6 we present the comparison between the results of numerical simulations and time traces obtained from two separate recordings. In subplots (a,b) we present the data collected from the first recording and in subplots (c,d) from the second one. Red lines in Fig. 6 represent the periodic attractor obtained numerically and black dots correspond to the experimental results. In subplots (a) and (c) we show time traces of the bell while subplots (b) and (d) are devoted to the clapper's behavior. Comparing the trajectories of the bell one can say that the numerical results show remarkable agreement with the experiment. Simultaneously, time traces of the clapper do not show such a convergence because the clapper of the examined bell perform non-periodic motion. The difference between the experimental data and the results of numerical simulation is mainly visible around the moments of impact.

The experiments reveal that each collision between the clapper and the bell has a bit different course. There is a simple explanation

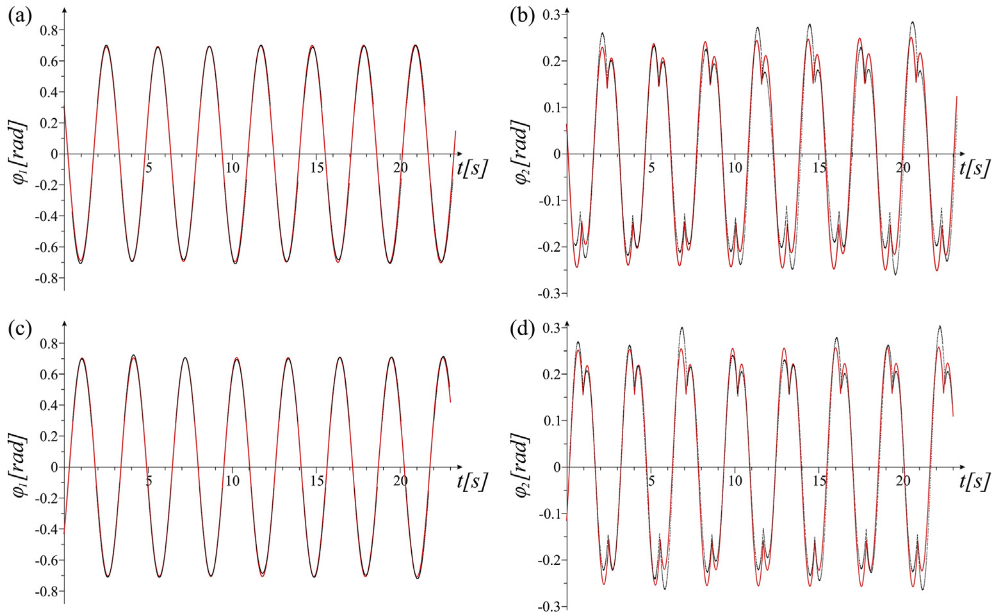

Fig. 6. Time traces showing the behavior of the bell (a,c) and the clapper (b,d) during normal working conditions. Subplots (a,b) correspond to the data obtained from the first recording and (c,d) from the second one. Black dots correspond to the data obtained experimentally while red lines present the results of numerical simulations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of this phenomenon. The bell starts to oscillate with a given frequency after the first contact with the clapper. The bell, as a musical instrument, is shaped so that the internal damping of these vibrations is small and in normal working conditions the clapper always encounters vibrating bell. Therefore, at the moment of the impact the bell's surface can be approaching or retreating from the clapper influencing the relative velocity of the striking points. To simulate this phenomenon the bell vibrations should be explicitly introduced to the model. In fact, to precisely describe the course of collisions one should use a finite-element model of the bell and its supporting structure which will extremely increase the complexity of the model, the computational power requirements and the time of calculations. Therefore, it will totally jeopardize the possibility of practical application.

Moreover, we have to remember that in most cases due to the lack of technical documentation we are not able to obtain precise values of the parameters that are crucial for the system's dynamics. Therefore, we should always consider whether expanding the model will significantly improve its reliability or just hinder its usage.

In Section 4 we prove experimentally that the discreet impact model presented in this paper is capable of simulating the effects of the clapper to the bell collisions when the bell does not vibrate. Also the value of restitution coefficient k has been determined experimentally in the idealized conditions [33]. In fact, the impact model itself is able to predict the angular velocities after each collision but it will require the changes of the restitution coefficient value. The problem of unrepeatable course of the clapper to the bell impacts has been earlier addressed [19] but to our best knowledge, up to now no one has shown how it affects the error of numerical simulations of the bells' and the clapper's dynamics.

To summarize, analyzing Fig. 6 one can say that the hybrid model introduced in this paper is able to simulate the bell's behavior with excellent accuracy and determine precisely the crucial features of the clapper's motion such as: the period of motion, the average amplitude of motion and to predict the moments of the clapper-bell collisions.

6. Conclusions

In this paper we present the hybrid dynamical model of the yoke-bell-clapper system. To increase its practical significance we expand the classical model with the description of the linear motor propulsion and energy dissipation. We determine the values of parameters included in the model basing on the measurements and experiments involving the bell in the Cathedral Basilica of St Stanislaus Kostka – *"The Heart of Lodz"*. After the detailed description of the model's evaluation we test its reliability by comparing the results of numerical simulations with experimentally obtained trajectories of the bell and the clapper.

First, we focus on the free motion conditions and try to validate the energy dissipation model. The presented results prove that assuming only viscous torsional damping in the bell's and the clapper's pin joints we are able to describe the effects of energy dissipation with acceptable accuracy. Hence, there is no need to introduce dry friction into the system. It will not bring a significant improvement and it will impede the tuning of the model by increasing the number of missing parameters.

To describe the clapper-bell impact we use the simple discreet model which employs the coefficient of energy restitution. We prove experimentally that the model enables to predict the effects of the single strike in the idealized conditions. Therefore, although the model does not describe the collision course it is thought to be adequate for the present purpose.

After validating the reliability of both "continuous" and "discreet" parts of the church bell model we try to examine the full model. For this purpose, using high speed camera recordings, we experimentally obtain time traces of the bell and the clapper in normal working conditions. Then we compare the data collected experimentally with the results of numerical simulations. Analyzing the bell's trajectories one can say that the proposed model precisely describes its behavior. Thanks to its reliability the model can be used by bell-hangers and engineers to determine time varying forces acting on the bell's support and induced by the bell's oscillations. Simultaneously, considering the clapper's trajectories we see that the accuracy of numerical simulations is limited. Using the described model we are able to determine crucial features such as the period and the average amplitude of the clapper's motion. We can also precisely predict the moments of impacts. The limitation of the model is visible in the vicinity of the clapper-bell collisions. As the result of numerical simulations we receive the periodic attractor with identical course of the successive impacts. Contrary, the experiments prove that each collision has a bit different run. This divergence between the experimental and numerical results can be simply explained.

In normal working conditions the bell vibrates incessantly. Hence, at the moment of impact, the bell's surface can be approaching or receding from the clapper. As a result, the angular velocity of the clapper after the impact can be enhanced or reduced respectively. To determine the effects of the strike using the introduced discreet model we have to know the relative velocities of the bell and the clapper just before the impact. As the proposed model does not involve the description of the bell's normal modes, we cannot precisely determine the relative velocities of the impacting points which slightly deteriorate the accuracy of the prediction.

The overall comparison between the results of numerical simulations and the data obtained experimentally leads to the conclusion that the presented hybrid model provides all the information about the system's dynamics which is crucial for bell-founders, bell-hangers and engineers working on bells or bell towers. Hence, it can be used both to develop mounting layouts for new bells and during the restoration processes. The biggest advantage of the presented hybrid model is that its simplicity and ease of use affect neither its reliability nor practical significance.

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